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Realization of absolute Galois groups as geometric fundamental groups

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Introduction

In the following, we are going to present a new tool in the study of absolute Galois groups.

Given a field F, it is of primary interest to classify its finite extensions. That is, given a separable closure $\bar{F}/F$, to understand the structure of the absolute Galois group $\mathrm{Gal}(\bar{F}/F)$. This is a very hard task. So much so that, at the present, a lot of problems remain open, even for the field $F = \mathbb{Q}$ of rational numbers.

We present a construction that translates this algebraic problem into a geometrical setting. Namely, we assign to each field F an Hausdorff, compact and connected topological space X_F, whose finite covering spaces are in one-to-one correspondence with finite field extensions of F. This is possible under the assumption that F contains $\mathbb{Q}(\mu_\infty)$: i.e. F is of characteristic zero and all roots of unity μ_∞ already belong to this field. We are going to prove the following.

Theorem (1.3.1)**.** *Let $F \supseteq \mathbb{Q}(\mu_\infty)$ be a field. There is a functor from the category of finite étale algebras over F to the category of finite covering spaces over X_F*

$$\textbf{\textit{FinÉtAlg}}/F \to \textbf{\textit{FinCov}}/X_F,$$

which is a category equivalence.

In particular, it follows that the étale fundamental group of X_F over a point $\chi \in X_F$ is isomorphic to the absolute Galois group of F: that is, we have an isomorphism

$$\pi^{\text{ét}}(X_F, \chi) \simeq \mathrm{Gal}(\bar{F}/F).$$

The construction is due to P. Scholze and R. Kucharczyk, who presented the idea in their recent paper [7]. This work is essentially a restructuring of their work, together with a record of some attempts to extend the limits of this construction. This thesis is not going to present a self-contained theory, but rather the pedant dissection of a mathematical object. As such, the techniques used will vary largely in complexity during the exposition. We aim to present every step in the most accessible way possible, trying to relegate harder techniques into the latest sections. Despite our efforts, a sound familiarity with algebraic tools is required.

In the first chapter, the Galois theories of finite extensions and topological covering spaces are going to be briefly introduced. That settled, we are going to present the main construction. We build the topological space $X_{\bar{F}}$ as the product of an infinite numbers of copies of the solenoid

$$\mathbb{Q}^{\vee} = \mathrm{Hom}(\mathbb{Q}, \mathbb{S}^1),$$

hence we obtain a compact Hausdorff connected topological space. We then discuss a natural action of the absolute Galois group G on $X_{\bar{F}}$, to then define the desired space X_F as the quotient $G\backslash X_{\bar{F}}$. The action being proper, X_F retains all topological properties of $X_{\bar{F}}$. In the last section the main theorem is proven.

In the second chapter, the topological space X_F is further studied. Namely, we concentrate on the relation between the number of path components of X_F and the multiplicative structure of F. We say that a field F is multiplicatively free if its multiplicative group $F^{\times}$ is free.

Theorem (2.1.6)**.** *If F is countable and all its finite extensions are multiplicatively free, then X_F is path connected.*

A field satisfying the hypotheses of the above theorem is $F = \mathbb{Q}(\mu_\infty)$.

We then compute sheaf cohomology groups of X_F with coefficients in a locally constant sheaf A. In particular, when A is a torsion abelian group we find out that cohomology groups are related to the cohomology of the absolute Galois group G; that is, there are isomorphisms

$$\mathrm{H}^p(X_F, A) \simeq \mathrm{H}^p(G, A) \qquad \forall p > 0.$$

Cohomology groups with integral coefficients do not admit such an explicit description, nonetheless they fit in short exact sequences

$$0 \to \mathrm{H}^p(X_F, \mathbb{Z}) \to \mathrm{H}^p(X_F, \mathbb{Q}) \to \mathrm{H}^p(X_F, \mathbb{Q}/\mathbb{Z}) \to 0 \qquad \forall p > 0.$$

In these sequences, we also know the central term, i.e. $\mathrm{H}^p(X_F, \mathbb{Q}) = \bigwedge^p \bar{F}^{\times} \otimes \mathbb{Q}$.

A discussion on the necessity of the hypothesis on roots of unity follows, in which we suggest a different construction which does not require said hypothesis: for every characteristic zero field F we build a topological space Z_F, which is compact and

Hausdorff but not connected. We then show that when Z_F is connected, it is homeomorphic to $X_{F(\mu_\infty)}$, hence this new construction ultimately reduces to the initial setting.

In the last chapter we investigate an analogous construction in the realm of algebraic geometry. That is, we present a connected complex scheme $\mathcal{X}_F$, whose finite étale covering maps are in correspondence with finite étale algebras over F.

Theorem (3.3.7)**.** *Let $F \supseteq \mathbb{Q}(\mu_\infty)$ be a field. There is a functor from the category of finite étale covers over* $\operatorname{Spec} F$ *to the category of finite étale covers over $\mathcal{X}_F$*

$$\textit{\textbf{FinÉtCov}}\,/\operatorname{Spec} F \to \textit{\textbf{FinÉtCov}}\,/\mathcal{X}_F$$

which is a category equivalence.

In particular, it follows that the étale fundamental group of $\mathcal{X}_F$ over a point $\bar{x} \in \mathcal{X}_F$ is isomorphic to the absolute Galois group of F: that is, we have an isomorphism

$$\pi^{\text{ét}}(\mathcal{X}_F, \bar{x}) \simeq \operatorname{Gal}(\bar{F}/F).$$

CHAPTER 1

Construction

This chapter aims to present in a crude and direct way the main construction. This construction, although coming from ideas in algebraic geometry, is elementary in nature and can be completely covered with some knowledge in topology and number theory. We aim to build a bridge from algebra to geometry, that will allow us to translate questions about field extensions to properties of topological covering spaces. The theories describing these objects already present striking similarities: we now present the key aspects of these theories, leaving the first chapters of [19] as a reference for proofs and details.

The theory of finite field extensions. Let F be a field and $\bar{F}/F$ a separable closure. For any finite Galois extension E/F, we consider the group $\mathrm{Gal}(E/F)$ of automorphisms of E as an F-algebra. The finite Galois extensions of F, connected by inclusion maps, form a filtered system whose colimit is $\bar{F}$. The corresponding Galois groups form an inverse system whose limit

$$\mathrm{Gal}(\bar{F}/F) = \varprojlim_E \mathrm{Gal}(E/F)$$

is called the absolute Galois group of F. The limit is intended as a limit of topological groups: given all finite Galois groups the discrete topology, the topology on the limit $\mathrm{Gal}(\bar{F}/F)$ is the coarsest to make all projections continuous. The resulting topological group is a profinite group.

Definition 1.0.1. A finite étale algebra E over F is a F-algebra that is isomorphic to the product of a finite number of finite separable extensions E_i/F:

$$E = E_1 \times \cdots \times E_r.$$

The absolute Galois group $G = \mathrm{Gal}(\bar{F}/F)$ is the key component in the classification of all finite étale algebras and, in particular, of all finite extensions of F. Consider the controvariant functor

$$\mathrm{Hom}_F(\bullet, \bar{F})\colon \mathbf{Fin\acute{E}tAlg}/F \to \mathbf{FinSet}$$

sending a finite étale algebra $E = E_1 \times \cdots \times E_r$ over F to the finite set of F-algebra homomorphism $E \to \bar{F}$; that is, the set

$$\mathrm{Hom}_F(E_1 \times \cdots \times E_r, \bar{F}) = \prod_{i=1}^{r} \mathrm{Hom}_F(E_i, \bar{F}).$$

The natural G-action on each E_i translates into a continuous action on the set of homomorphisms and the following result holds.

Theorem 1.0.2 ([19, Theorem 1.5.4])**.** *The just defined functor*

$$\mathrm{Hom}_F(\bullet, \bar{F})\colon \boldsymbol{Fin\acute{E}tAlg}/F \to \mathrm{Gal}(\bar{F}/F)\text{-}\boldsymbol{FinSet}$$

is an anti-equivalence between the category of finite étale algebras over F to the category of finite sets equipped with a continuous G-action.

An application of the Yoneda Lemma shows that the group of automorphism of the functor above is

$$\mathrm{Hom}_F(\bar{F}, \bar{F}) = \mathrm{Gal}(\bar{F}/F),$$

the absolute Galois group of F.

The theory of topological covering spaces. Let X be a connected topological space.

Definition 1.0.3. A continuous map of topological spaces $p\colon Y \to X$ is a trivial finite covering if there is a finite discrete set D and a homeomorphism $X \times D \to Y$ making the following diagram commute

$$\begin{tikzcd} X \times D \arrow[r] \arrow[dr] & Y \arrow[d, "p"] \\ & X; \end{tikzcd}$$

more generally, $p\colon Y \to X$ is a finite covering if every point in X has an open neighbourhood $U \subseteq X$ such that the restriction $p\colon p^{-1}(U) \to U$ is a trivial finite covering.

A morphism of covering spaces $Y_1, Y_2 \to X$ is a continuous map $Y_1 \to Y_2$ which commutes with the covering maps; if both Y_1, Y_2 are connected, this morphism is a covering map itself. In particular we can consider a connected finite cover $p\colon Y \to X$ and the group $\mathrm{Aut}(Y|X)$ of cover automorphims of Y; that is the homeomorphisms $Y \to Y$ that commute with p. We call p a Galois cover if the continuous map induced by p on the quotient

$$\mathrm{Aut}(Y|X)\backslash Y \to X$$

is an homeomorphism.

Let $Y \to X$ be a Galois cover. The group $\mathrm{Aut}(Y|X)$ determines all intermediate finite connected covers $Y \to Z \to X$. That is, given said tower, $Y \to Z$ is a Galois cover whose group $\mathrm{Aut}(Y|Z)$ is naturally identified with a subgroup $H < \mathrm{Aut}(Y|X)$; in this case, the quotient cover $H\backslash Y \to X$ is isomorphic to $Z \to X$. Vice versa, every subgroup $H < \mathrm{Aut}(Y|X)$ determines an intermediate cover $Y \to X$, whose corresponding subgroup of $\mathrm{Aut}(Y|X)$ is H itself.

In analogy with the previous theory, we may fix a point $x \in X$ and consider the functor

$$Fib_x\colon \mathbf{FinCov}/X \to \mathbf{FinSet}$$

that sends a cover $p\colon Y \to X$ to the fiber $p^{-1}(x)$, which is a discrete finite set. The key object for the classification of the topological covering spaces of X is the étale fundamental group of X in x, which is defined as the group of automorphisms of the fiber functor:

$$\pi_1^{\text{ét}}(X,x) = \mathrm{Aut}\,(Fib_x)\,.$$

If we fix a Galois cover $Y \to X$ and restrict the fiber functor to the full subcategory of finite covers of X that are quotient of Y, we obtain a finite group $\mathrm{Aut}(Fib_x^Y) = \mathrm{Aut}(Y|X)$. The inverse system of Galois covers gives an inverse system of the corresponding automorphisms group, whose limit coincide, as an abstract group, with the étale fundamental group:

$$\pi_1^{\text{ét}}(X,x) = \mathrm{Aut}\,(Fib_x) = \varprojlim_Y \mathrm{Aut}(Fib_x^Y) = \varprojlim_Y \mathrm{Aut}(Y|X). \tag{1.1}$$

This gives the fundamental group a profinite topology.

Theorem 1.0.4. *The fiber functor splits through an equivalence*

$$Fib_x\colon \boldsymbol{FinCov}/X \to \pi_1^{\acute{e}t}(X,x)\text{-}\boldsymbol{FinSet}$$

between the category of finite covers over X to the category of finite sets equipped with a continuous action of the étale fundamental group $\pi_1^{\acute{e}t}(X,x)$.

When X is not only connected, but also path connected and semi-locally simply connected, the theorem is well-know and the étale fundamental group $\pi_1^{\text{ét}}(X,x)$ coincide withe the profinite completion of the usual fundamental group $\pi_1(X,x)$ of the space [19, Corollary 2.3.9].

This is enough to come up with a genuinely interesting question, which the present chapter is going to answer.

Question. Given a field F, can we find a topological space X_F whose finite covers are in correspondence with finite extensions of our base field F? That is, a space that comes equipped with an anti-equivalence

$$\mathbf{Fin\acute{E}tAlg}\,/F \to \mathbf{FinCov}\,/X_F \quad ? \tag{1.2}$$

One may immediately think the solution lies in a clever use of classifying spaces, but the question is trickier than it looks. The main difficulty is hidden in the necessary step of profinite completion: although it is not hard to build a topological space with a fixed fundamental group, is not clear how to choose one that behaves as desired under completion. We are thus going to take a different approach, which is not only feasible but, as a nice perk, will produce a much more concrete space to study.

1.1 The solenoid

In this section, we present a curious topological group that plays a fundamental role in our construction. It is known in literature as the *solenoid*. One can define the solenoid $\mathcal{S}$ as the Pontryagin dual of the discrete group of rational numbers $\mathbb{Q}^\vee$. Concretely, you take the group of rational numbers, equip it with the discrete topology, and consider the set of group homomorphisms into the circle

$$\mathbb{Q}^\vee = \mathrm{Hom}(\mathbb{Q}, \mathbb{S}^1),$$

equipped with the compact-open topology. A base of said topology is given by the family of subsets

$$V(K,U) = \{f \in \mathrm{Hom}(\mathbb{Q}, \mathbb{S}^1) \mid f(K) \subseteq U\},$$

where $K \subseteq \mathbb{Q}$ varies among all compact subspaces and $U \subseteq \mathbb{S}^1$ among all open ones. With said topology and composition as the operation, $\mathbb{Q}^\vee$ turns into a topological group [5, Theorem 23.15].

Notice that is not enough to choose where to send 1 to determine a group homomorphism $\mathbb{Q} \to \mathbb{S}^1$: sending 1 to a fraction of the full circle $\alpha \in \mathbb{S}^1$ leaves open some choices about where to send $1/N$, for every integer N; to be precise, exactly N choices which differs by an N-th of the full angle. These choices are not independent and it is easy to convince yourself that an uncountable number of choices is required! We thus expect this space not to be a copy of the circle, but a quite “bigger ”space. Nonetheless, it shares with the circle some nice topological properties.

Proposition 1.1.1. *The topological space $\mathcal{S}$ is Hausdorff, compact and connected.*

Proof. We can think at the rational numbers $\mathbb{Q}$ as a direct limit of rank 1 free subgroups:

$$\mathbb{Q} = \varinjlim_n \tfrac{1}{n}\mathbb{Z}$$

where the limit is taken over the poset of positive integers ordered by divisibility. The solenoid is thus a limit over the opposite poset:

$$\mathcal{S} = \mathbb{Q}^\vee = \varprojlim_n \mathrm{Hom}(\tfrac{1}{n}\mathbb{Z}, \mathbb{S}^1).$$

These duals are much easier to compute: since any homomorphism is determined by the image of 1, all objects in the limit coincide with the circle! That is, our space is a limit of circles

$$\mathcal{S} = \varprojlim_n \mathbb{S}^1_{(n)}$$

where the map $\mathbb{S}^1_{(mn)} \to \mathbb{S}^1_{(n)}$ is the m-th power map. Limit of Hausdorff, compact connected spaces is again Hausdorff, compact and conencted [3, theorem 6.1.20]. □

This space comes with a weird pathology: it is connected but not path-connected. It has, in fact, an infinite (uncountable) number of path components! We are going to present a useful proposition to compute the number of path components.

Proposition 1.1.2. *Let M be a torsion free discrete abelian group. We have a natural isomorphism*

$$\pi_0(M^\vee) \simeq \mathrm{Ext}^1(M, \mathbb{Z}).$$

Proof. From the short exact sequence of topological groups $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{S}^1 \to 0$, we get

$$\mathrm{Hom}(M, \mathbb{R}) \to \mathrm{Hom}(M, \mathbb{S}^1) \to \mathrm{Ext}^1(M, \mathbb{Z}) \to 0;$$

where the last term vanishes because $\mathbb{R}$ is divisible, and thus have trivial cohomology. We are going to show that $\mathrm{Hom}(M, \mathbb{R})$ is mapped to the path component of the origin of $M^\vee$.

The space $\mathrm{Hom}(M, \mathbb{R})$ is path connected: given any point $\varphi : M \to \mathbb{R}$, the path $\gamma(t, m) = t \cdot \varphi(m)$ connects it to the origin. The continuous map from the exact sequence thus sends this space into the path component of 0. We claim surjectivity.

Let x be a point in said path component and choose a path γ connecting it to the origin. For every $m \in M$, we have a path $t \mapsto \gamma(t, m)$ in $\mathbb{S}^1$. We can lift all these paths to the universal covering space

$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ M \arrow[ur, dashed, "\tilde{\gamma}(-,m)"] \arrow[r, "\gamma(-,m)"'] & \mathbb{S}^1. \end{tikzcd}$$

In order for $\tilde{\gamma}(t, -)$ to be a path in $\mathrm{Hom}(M, \mathbb{R})$, at every fixed time t the map $m \mapsto \tilde{\gamma}(t, m)$ has to be a group homomorphism; let's show that is the case. Consider

$$t \mapsto \tilde{\gamma}(t, m) + \tilde{\gamma}(t, n) - \tilde{\gamma}(t, m + n).$$

Since the projection onto $\mathbb{S}^1$ is trivial by hypothesis, this has to be a continuous map into the discrete subgroup of the integers $\mathbb{Z}$ in $\mathbb{R}$, thus constant. Also, for $t = 0$ we know it vanishes. □

Whit this result, we can compute the number of path components of the solenoid.

Theorem 1.1.3. *The topological space $\mathcal{S}$ is not path connected. In particular, $\mathcal{S}$ has an uncountable number of connected components.*

Proof. From the previous proposition 1.1.2 we have an isomorphism

$$\pi_0(\mathcal{S}) = \mathrm{Ext}^1(\mathbb{Q}, \mathbb{Z}).$$

We need to show that the latter is an uncountable abelian group! In order to see that, consider the injective resolution of $\mathbb{Z}$

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0,$$

where both $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are divisible, hence injective. We can use this resolution to compute $\mathrm{Ext}^1(\mathbb{Q}, \mathbb{Z})$, applying $\mathrm{Hom}(\mathbb{Q}, -)$ and taking cohomology; doing so, we find that $\mathrm{Ext}^1(\mathbb{Q}, \mathbb{Z})$ is the cokernel of the map

$$\mathrm{Hom}(\mathbb{Q}, \mathbb{Q}) \to \mathrm{Hom}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}).$$

The first group is isomorphic to $\mathbb{Q}$, while the latter is uncountable:

$$\mathrm{Hom}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) = \varprojlim_n \mathrm{Hom}(\tfrac{1}{n}\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \varprojlim_n \mathbb{Q}/\mathbb{Z},$$

where transition maps are multiplication by the index. We can explicitly construct this limit as a subgroup

$$\{(a_n)_n \in \prod_{\mathbb{N}_{>0}} \mathbb{Q}/\mathbb{Z} \mid a_n = m \cdot a_{nm} \ \forall n, m \in \mathbb{N}_{>0}\},$$

and observe we obtain, in fact, an uncountable group. □

Notice that, from the proof of Proposition 1.1.2, we actually get more information: namely, the path-component of the identity is isomorphic to $\mathrm{Hom}(\mathbb{Q}, \mathbb{R}) \simeq \mathbb{R}$. Therefore the fundamental group at 0 of the solenoid is trivial: $\pi_1(\mathcal{S}, 0) = 0$. We would love to use this data to conclude that there are no covering spaces but, as $\mathcal{S}$ is not path connected, the usual theory of covering spaces does not apply! In fact, the solenoid does have covering spaces of infinite degree, but, as we are now going to show, not of finite degree.

Theorem 1.1.4. *The solenoid $\mathcal{S}$ has no non-trivial finite covering space.*

To prove this theorem, we need a strategic lemma that will allow us to exploit the compactness hypothesis.

Lemma 1.1.5 (Compactness Argument)**.** *Let $X = \varprojlim_\alpha X_\alpha$ the limit of a co-filtered system of compact Hausdorff spaces. Let $\pi\colon Y \to X$ be a finite covering map. There exists an index α_0 and a finite covering map $\pi_0\colon Y_0 \to X_0$ that fits in a pull-back diagram*

$$\begin{array}{ccc} Y & \longrightarrow & Y_0 \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi_0} \\ X & \longrightarrow & X_0. \end{array}$$

Proof. We first have to recall that the topology on X is defined via the base of sets $U \subseteq X$ that are pre-image of some open set $U_\alpha \subseteq X_\alpha$ from somewhere in the inverse system. We can restrict our attention to those open sets of this base that trivialize the covering map π. Because X is compact, we can choose a finite number of them forming an open cover, say $U_1, \ldots, U_n$.

We can now think at our covering space as a collage: we can assemble Y gluing together d copies of each open set U_i. Precisely, there exist finite sets $D_1, \ldots, D_n$ and continuous functions

$$\phi_{ij} : U_{ij} = U_i \cap U_j \to \{f\colon D_i \to D_j \mid f \text{ is bijective}\},$$

where the codomain is intended as a discrete set, which we can think as the instructions to glue together Y. These have to satisfy a cocycle condition $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ for all $u \in U_{ijk} = U_i \cap U_j \cap U_k$. Because of our clever choice of the trivializing open sets U_i, we can find an index β such that all U_i are pre-image of some open sets of X_1!

We need to choose another finite open cover $V_1, \ldots, V_m \subseteq X$ of sets that are both

1. pre-image of some open set $V_\alpha \subseteq X_\alpha$ from somewhere in the inverse system,
2. contained in some intersection U_{ij} (hence trivializing),

3. all instructions ϕ_{ij}, restricted to V are constant.

There is no problem in doing that since all ϕ_{ij} are locally constant. Now, we find an object X_0 in the inverse system such that all V_i and all U_j are pre-image of some open subset of X_0, and glue together a covering space $Y_0 \to X_0$. This covering map pulls back to π by construction. $\square$

This argument is a very fundamental step in our construction. We are going to discuss later why the compactness hypothesis is so important for us.

Proof of theorem 1.1.4. Let $Y \to \mathcal{S}$ a finite connected covering space. From the description as limit $\mathcal{S} = \varprojlim_n \mathbb{S}^1_{(n)}$ and the compactness argument 1.1.5 we find an index k and a finite cover $Y' \to \mathbb{S}^1_{(k)}$ that fits in a pull-back diagram

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ \mathcal{S} & \longrightarrow & \mathbb{S}^1_{(k)}. \end{array}$$

We know all finite covering maps into the circle: these are all m-th power maps $\mathbb{S}^1 \to \mathbb{S}^1$, $x \mapsto x^m$. Said otherwise, the previous diagram can be written as

$$\begin{array}{ccc} Y & \longrightarrow & \mathrm{Hom}(\frac{1}{mk}\mathbb{Z}, \mathbb{S}^1) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\mathbb{Q}, \mathbb{S}^1) & \longrightarrow & \mathrm{Hom}(\frac{1}{k}\mathbb{Z}, \mathbb{S}^1), \end{array}$$

where all maps are induced by natural inclusions. Notice that, in order to compute this pull-back, we can take duals. That is, Y is the dual of the push-out of the dual diagram; thus the thesis follows once we convince ourselves that the following is a push-out diagram:

$$\begin{array}{ccc} \frac{1}{k}\mathbb{Z} & \longrightarrow & \frac{1}{mk}\mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{Q} & = & \mathbb{Q}. \end{array}$$

This is exactly to say the map $\mathbb{Z} \to \mathbb{Z}$ given by multiplication by m, tensored by $\mathbb{Q}$ gives an isomorphism. Hence $Y \to \mathcal{S}$ is trivial. $\square$

The list of properties of the solenoid is far from over and could go on for a while. We content ourselves with the properties we are going to need in the following discussion. In this last proposition, we aim to underline the arithmetic nature of this object.

Let $\mathbb{A}_\mathbb{Q}$ be the ring of adeles over the field of rational numbers $\mathbb{Q}$. We define the ring of finite adeles $\mathbb{A}_\mathbb{Q}^{\text{fin}}$ as the restricted product [16, Chapter 5.1] of all completions of $\mathbb{Q}$ over their ring of integers, that is all p-adic fields $\mathbb{Q}_p$ with respect to their ring of integers $\mathbb{Z}_p \subseteq \mathbb{Q}_p$. This is the subring

$$\mathbb{A}_\mathbb{Q}^{\text{fin}} = \{(a_p)_p \in \textstyle\prod_p \mathbb{Q}_p \mid a_p \in \mathbb{Z}_p \text{ for all but finitely many } p\}.$$

With the topology we get declaring open all subgroups of the form

$$\prod_{p \in S} \mathbb{Q}_p \times \prod_{q \notin S} \mathbb{Z}_q,$$

for every finite set of primes S, $\mathbb{A}_\mathbb{Q}^{\text{fin}}$ is a topological ring. We define $\mathbb{A}_\mathbb{Q} = \mathbb{R} \times \mathbb{A}_\mathbb{Q}^{\text{fin}}$ and get a topological ring, with the product topology. Both $\mathbb{R}$ and $\mathbb{A}_\mathbb{Q}^{\text{fin}}$ are closed additive subgroups of $\mathbb{A}_\mathbb{Q}$. The diagonal embedding $\mathbb{Q} \hookrightarrow \mathbb{A}_\mathbb{Q}$ has discrete image [16, Theorem 5.11]. The adele ring is strictly related to the solenoid $\mathcal{S}$.

Lemma 1.1.6. *There is an isomorphism of topological groups*

$$\mathcal{S} \simeq \mathbb{A}_\mathbb{Q}/\mathbb{Q}.$$

Also, the injective map $\operatorname{Hom}(\mathbb{Q}, \mu_\infty) \hookrightarrow \operatorname{Hom}(\mathbb{Q}, \mathbb{S}^1)$ *induced by* $\iota\colon \mu_\infty \hookrightarrow \mathbb{S}^1$ *corresponds to the inclusion*

$$\mathbb{A}_\mathbb{Q}^{\mathit{fin}} \hookrightarrow \mathbb{A}_\mathbb{Q}/\mathbb{Q},$$

the isomorphic image of the second factor through the projection $\mathbb{A}_\mathbb{Q} \to \mathbb{A}_\mathbb{Q}/\mathbb{Q}$. *Furthermore, the injective map* $\operatorname{Hom}(\mathbb{Q}/\mathbb{Z}, \mu_\infty) \hookrightarrow \operatorname{Hom}(\mathbb{Q}, \mu_\infty)$ *induced by* $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ *corresponds to the natural inclusion* $\hat{\mathbb{Z}} \subseteq \mathbb{A}_\mathbb{Q}^{\mathit{fin}}$.

Proof. The statement follows writing in a different way the limit (1.1) we presented in the proof of Proposition (1.1.1): writing $\mathbb{Q} = \varinjlim_n \frac{1}{n}\mathbb{Z}$, we get

$$\mathcal{S} = \varprojlim_n \operatorname{Hom}(\tfrac{1}{n}\mathbb{Z}, \mathbb{S}^1) = \varprojlim_n \operatorname{Hom}(\tfrac{1}{n}\mathbb{Z}, \mathbb{R}/\mathbb{Z}) = \varprojlim_n \mathbb{R}/n\mathbb{Z}, \tag{1.3}$$

where the maps in the last limit are induced by the identity $\mathbb{R} \to \mathbb{R}$. Consider the copy of the group of profinite integers

$$\hat{\mathbb{Z}} \subseteq \mathbb{A}_\mathbb{Q}^{\text{fin}} = 0 \times \mathbb{A}_\mathbb{Q}^{\text{fin}} \hookrightarrow \mathbb{A}_\mathbb{Q}.$$

For every integer n, from the closed group embedding of the first factor $\mathbb{R} = \mathbb{R} \times 0 \hookrightarrow \mathbb{A}_\mathbb{Q}$, $x \mapsto (x, 0)$ we get the continuous and closed group homomorphism

$$\varphi_n\colon \frac{\mathbb{R}}{n\mathbb{Z}} \to \frac{\mathbb{A}_\mathbb{Q}}{\mathbb{Q} + n\hat{\mathbb{Z}}},$$

which we claim to be an isomorphism. First, notice the map is well defined, since for every $x \in n\mathbb{Z}$ we have

$$x \mapsto (x, (0)_p) = (x, (x)_p) + (0, (-x)_p) \in \mathbb{Q} + n\hat{\mathbb{Z}}.$$

That φ_n is injective is clear: an element in the kernel of $\mathbb{R} \to \mathbb{A}_{\mathbb{Q}}/\mathbb{Q} + n\hat{\mathbb{Z}}$ is a real number x such that $x \in \mathbb{Q} \cap n\hat{\mathbb{Z}} = n\mathbb{Z}$. Surjectivity follows from the strong approximation theorem: the cokernel $\mathbb{A}_{\mathbb{Q}}/\mathbb{R} + \mathbb{Q} + n\hat{\mathbb{Z}}$ is trivial because $\mathbb{R} + \mathbb{Q} + n\hat{\mathbb{Z}}$ generates all $\mathbb{A}_{\mathbb{Q}}$ [16, Corollary 5.9]. These φ_n are compatible with the maps in the projective system of limit (1.3), giving us the isomorphism

$$\varprojlim_n \mathbb{R}/n\mathbb{Z} = \varprojlim_n \mathbb{A}_{\mathbb{Q}}/\mathbb{Q} + n\hat{\mathbb{Z}} = \mathbb{A}_{\mathbb{Q}}/\mathbb{Q},$$

where the last equality follows from $\mathbb{A}_{\mathbb{Q}}$ being complete and $n\hat{\mathbb{Z}}$ being a system of open subgroups with empty intersection.

For the second statement, notice we are interested in the subgroup

$$\varprojlim_n \mathbb{Q}/n\mathbb{Z} \hookrightarrow \varprojlim_n \mathbb{R}/n\mathbb{Z} = \mathbb{A}_{\mathbb{Q}}/\mathbb{Q},$$

where every n-th component map is induced by the same lift of $\iota\colon \mu_\infty = \mathbb{Q}/\mathbb{Z} \to \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ to an inclusion $\mathbb{Q} \subseteq \mathbb{R}$. Composing $\mathbb{Q} \subseteq \mathbb{R}$ with φ_n one gets a map

$$\psi_n\colon \frac{\mathbb{Q}}{n\mathbb{Z}} \to \frac{\mathbb{A}_{\mathbb{Q}}}{\mathbb{Q} + n\hat{\mathbb{Z}}}$$

that factors through $\mathbb{A}_{\mathbb{Q}}^{\text{fin}}/n\hat{\mathbb{Z}}$, since $(x, 0) = (0, (-x)_p)$ in $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$. This shows also that the image of ψ_n coincides with the image of the diagonal embedding $\mathbb{Q} \hookrightarrow \mathbb{A}_{\mathbb{Q}}^{\text{fin}} \to \mathbb{A}_{\mathbb{Q}}^{\text{fin}}/n\hat{\mathbb{Z}}$. This proves the map is surjective, since $\mathbb{Q} + n\hat{\mathbb{Z}}$ generates $\mathbb{A}_{\mathbb{Q}}^{\text{fin}}$ because of the strong approximation theorem [1, Chapter II, 15], thus

$$\varprojlim_n \mathbb{Q}/n\mathbb{Z} = \varprojlim_n \mathbb{A}_{\mathbb{Q}}^{\text{fin}}/n\hat{\mathbb{Z}} = \mathbb{A}_{\mathbb{Q}}^{\text{fin}}.$$

Finally, notice that through the identifications of (1.3), we can write the last inclusion as

$$\operatorname{Hom}(\mathbb{Q}/\mathbb{Z}, \mu_\infty) = \varprojlim_n \operatorname{Hom}(\tfrac{1}{n}\mathbb{Z}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \hookrightarrow \varprojlim_n \operatorname{Hom}(\tfrac{1}{n}\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \operatorname{Hom}(\mathbb{Q}, \mu_\infty),$$

where the n-th map is $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Q}/n\mathbb{Z}$, the quotient of the natural inclusion $\mathbb{Z} \subseteq \mathbb{Q}$. The image in $\mathbb{A}_{\mathbb{Q}}^{\text{fin}}/n\hat{\mathbb{Z}}$ is therefore the copy of the integers through the diagonal embedding $\mathbb{Z} \subseteq \mathbb{Q} \hookrightarrow \mathbb{A}_{\mathbb{Q}}^{\text{fin}} \to \mathbb{A}_{\mathbb{Q}}^{\text{fin}}/n\hat{\mathbb{Z}}$. Computing the inverse limit we get

$$\varprojlim_n \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}} \subseteq \mathbb{A}_{\mathbb{Q}}^{\text{fin}}.$$

□

1.2 Main construction

Let's fix a base field F of characteristic 0 that contains all roots of unity μ_∞. Fix an algebraic closure $\bar{F}/F$ and call $G = \mathrm{Gal}(\bar{F}/F)$ the corresponding absolute Galois group. From this point on, these are going to be our assumptions. The hypothesis on the roots of unity is quite restrictive, we are going to dedicate a section of the next chapter to its discussion.

Equip the multiplicative group $\bar{F}^\times$ with the discrete topology and consider its Potryagin dual (the construction we had for $\mathbb{Q}$ in the previous section): consider the set

$$\mathrm{Hom}(\bar{F}^\times, \mathbb{S}^1),$$

equipped with the compact-open topology. This is a topological group. Let $\iota\colon \mu_\infty \hookrightarrow \mathbb{S}^1$ be your favorite injective homomorphism and consider the subspace of characters that agree with said embedding

$$X_{\bar{F}} = \{\chi \in \mathrm{Hom}(\bar{F}^\times, \mathbb{S}^1) \mid \chi_{|\mu_\infty} \equiv \iota\colon \mu_\infty \to \mathbb{S}^1\}.$$

This space comes with a natural continuous left action of the absolute Galois group G, given by pre-composition with the inverse: $\sigma \cdot \chi(x) = \chi(\sigma^{-1}(x))$. This is not the space we were looking for, having no non-trivial finite covering spaces, but we are almost there. In our construction $X_{\bar{F}}$ has the role of a universal covering space, although not exactly being one. The space with the desired properties is the quotient

$$X_F = G \backslash X_{\bar{F}}.$$

Let's start by showing that $X_{\bar{F}}$ has some nice geometrical properties.

Theorem 1.2.1. *The topological space $X_{\bar{F}}$ is non-empty, Hausdorff, compact and connected.*

Proof. The multiplicative group $\bar{F}^\times$ is divisible: that is, every number $x \in \bar{F}^\times$ has an n-th root. Its torsion free quotient $\bar{F}^\times_{\mathrm{tf}} = \bar{F}^\times/\mu_\infty$ is thus a torsion-free abelian group that is uniquely divisible, i.e. multiplication by rational numbers is defined: a $\mathbb{Q}$-vector space. The exact sequence

$$0 \to \mu_\infty \to \bar{F}^\times \to \bar{F}^\times_{\mathrm{tf}} \to 0$$

splits, because μ_∞ is divisible, thus injective. Hence we can write $\bar{F}^\times = \bar{F}^\times_{\mathrm{tf}} \oplus \mu_\infty$ and consequently the isomorphism

$$\mathrm{Hom}(\bar{F}^\times, \mathbb{S}^1) = \mathrm{Hom}(\bar{F}^\times_{\mathrm{tf}}, \mathbb{S}^1) \times \mathrm{Hom}(\mu_\infty, \mathbb{S}^1)$$

as topological groups. We now see how our space $X_{\bar{F}}$ as a coset of the subgroup $\mathrm{Hom}(\bar{F}_{\mathrm{tf}}^{\times}, \mathbb{S}^1)$ of characters that are trivial on all roots of unity. In particular, we have a homeomorphism

$$X_{\bar{F}} \simeq \mathrm{Hom}(\bar{F}_{\mathrm{tf}}^{\times}, \mathbb{S}^1).$$

The latter space is much easier to study (but would be much harder to define a nice G action on it). We fix a basis I for $\bar{F}_{\mathrm{tf}}^{\times}$ over $\mathbb{Q}$ and we write this space as a product of infinitely many copies of the solenoid:

$$\mathrm{Hom}(\bar{F}_{\mathrm{tf}}^{\times}, \mathbb{S}^1) = \mathrm{Hom}\left(\bigoplus_{i \in I} \mathbb{Q}, \mathbb{S}^1\right) = \prod_{i \in I} \mathrm{Hom}(\mathbb{Q}, \mathbb{S}^1) = \prod_{i \in I} \mathcal{S}.$$

The claim follows, the solenoid being nonempty, Hausdorff, connected and compact. □

In the previous proof, we got an idea of how horrendous is the space we are working with. It has to be connected, but not path-connected, like the solenoid, and, at the same time, it must be, in some sense, infinite-dimensional, unlike the solenoid which we think, in some sense, one dimensional. This allows us to deduce some other geometric properties of this space, his étale fundamental group for example.

Corollary 1.2.2. *The space $X_{\bar{F}}$ has no non-trivial finite covering space.*

Proof. Let $\Delta\colon \mathcal{S} \to \prod_{i \in I} \mathcal{S}$ the diagonal embedding and $p\colon A \to \prod_{i \in I} \mathcal{S}$ be a finite covering map. The pullback

$$\begin{array}{ccc} \Delta_* A & \longrightarrow & A \\ \downarrow & & \downarrow p \\ \mathcal{S} & \xrightarrow{\Delta} & \prod_{i \in I} \mathcal{S} \end{array}$$

is a finite covering map on the solenoid of the same degree. According to Theorem 1.1.4 the solenoid has no non-trivial covering space of finite degree, therefore p has to be trivial as well. □

We are left to investigate the Galois action. We wish for the action to be enough well behaved to produce us covering spaces when we take quotients. It would be enough for the action to be properly discontinuous and free, but in this case, a bit more is true, so that our quotient spaces are going to maintain the Hausdorff property as well.

Theorem 1.2.3. *The action of G on $X_{\bar{F}}$ is proper and free.*

Unlike the above discussion, this aspect is not geometric in nature, but algebraic! To understand this action, we need to exploit the definition, back to the natural action of G on the field $\bar{F}$. The content we are seeking could not come from anything but a number theory lemma.

Lemma 1.2.4. *Let F be a field of characteristic 0, not necessarly containing μ_∞, and let $\bar{F}/F$ be an algebraic closure. The corresponding Galois group $G = \mathrm{Gal}(\bar{F}/F)$ operates freely on the set*

$$\mathcal{J}(\bar{F}, k) = \{\chi \colon \bar{F}^\times \to k^\times \mid \chi \textit{ is a group homorphism, injective on } \mu_\infty\},$$

for any field k.

Proof of the Lemma. Let $\sigma \in G$ a nontrivial element and $\chi \in \mathcal{J}(\bar{F}, k)$ a character. We need to show that ${}^\sigma\chi \neq \chi$. Our goal is to find an element $y \in \bar{F}^\times$ such that $\sigma(y)/y$ is a non-trivial root of unity ζ; we could then conclude that

$$1 \neq \chi(\zeta) = \chi(\sigma(y)/y),$$

hence that $\chi(\sigma(y)) \neq \chi(y)$ and, therefore, that the action is free.

First things first, let's get rid of everything on which σ acts trivially: upon replacing F by the fixed field of the closure of $\langle\sigma\rangle$, we can assume that σ is a topological generator of G (which, in turn, is now a pro-cyclic group). Notice that, as claimed, there is no element of $\bar{F} \setminus F$ on which σ acts trivially: take $\alpha \in \bar{F} \setminus F$ and let E/F be the Galois closure of $F(\alpha)/F$; the group $\mathrm{Gal}(E/F)$ is a nontrivial quotient of G by a closed subgroup of finite index (that is $\mathrm{Gal}(\bar{F}/E)$), which is therefore open. The dense subgroup generated by σ intersects the open cosets of this group, thus there exists an integer n such that σ^n acts non-trivially on α. Hence σ cannot fix α.

On the previous assumption, the base field F cannot be closed by radicals: there exist a prime p such that the p-th power map $F^\times \to F^\times$ is not surjective. For sake of contradiction, assume the contrary. Then, F must at least contain all roots of unity and G has to act trivially on them. From the short exact sequence $0 \to \mu_p \to \bar{F}^\times \to \bar{F}^\times \to 0$ associated to the p-th power map, we get the exact sequence in cohomology

$$F^\times \longrightarrow F^\times \longrightarrow \mathrm{H}^1(G,\, \mu_p) \longrightarrow 0,$$

where the last term vanishes by Hilbert90. The cohomology group, in this case, is easily described: G is acting trivially on μ_p, hence $\mathrm{H}^1(G,\, \mu_p) \simeq \mathrm{Hom}(G,\, \mathbb{Z}/p\mathbb{Z})$. But G, being pro-cyclic, must have some non-trivial cyclic quotient, thus $\mathrm{Hom}(G,\, \mathbb{Z}/p\mathbb{Z})$ cannot be trivial for every p.

Therefore there exists a prime p and a number $a \in F$ such that $x^p - a$ is irreducible in F. Picking a solution y in $\bar{F}$, we must have that $\sigma(y)/y \neq 1$ is a p-th root of unity, as desired. □

Proof of the theorem. The action being proper is equivalent to the map

$$\begin{aligned} G \times X &\to X \times X \\ (g, x) &\mapsto (gx, x) \end{aligned}$$

being proper, which follows from compactness of all spaces involved. The action being free follows from Lemma 1.2.4, choosing the target field k to be the complex numbers $\mathbb{C}$, where we identify $\mathbb{S}^1$ with the unit circle. □

The aforementioned quotient $X_F = G \backslash X_{\bar{F}}$ is therefore a non-empty, Hausdorff, connected and compact topological space.

1.3 Galois correspondence

We built a nice topological space $X_{\bar{F}}$, equipped with a natural action of G that is proper and free. We can therefore consider the quotient space $X_F = G \backslash X_{\bar{F}}$, which we know, from the properties of the G-action, to be Hausdorff, compact and connected. This is the space we were seeking: in this section, we are going to prove that we have a correspondence between finite field extensions E/F and finite covers of X_F.

For any finite field extension E/F, the same construction can be carried on. The extension $\bar{F}/E$ is an algebraic closure as well and the action of the subgroup $\mathrm{Gal}(\bar{F}/E) < G$ on $X_{\bar{F}}$ is proper and free too, therefore the topological space

$$X_E = \mathrm{Gal}(\bar{F}/E) \backslash X_{\bar{F}}$$

is Hausdorff, compact and connected. By definition, it comes equipped with a finite covering map $X_E \to X_F$ whose degree is equal to the index of the subgroup $\mathrm{Gal}(\bar{F}/E)$ in G.

We now wish to extend this construction to a functor $\mathbf{Fin\acute{E}tAlg}/F \to \mathbf{FinCov}/X_F$. Given a finite étale algebra $E_1 \times \cdots \times E_r$ over F, the natural finite covering of X_F to consider is the disjoint union

$$E_1 \times \cdots \times E_r \mapsto X_{E_1} \sqcup \cdots \sqcup X_{E_r}.$$

The image of maps through this functor is equally straightforward to define: every field homomorphism $E_1 \to E_2$ over F is either trivial or an inclusion; for which we already know how to get a covering map. The general case reduces to the case of fields, just notice that a morphism between finite étale algebras over F breaks into components

$$\mathrm{Hom}_F(E_1 \times \cdots \times E_r, E'_1 \times \cdots \times E'_s) = \coprod_{i=1}^{r} \prod_{j=1}^{s} \mathrm{Hom}_F(E_i, E'_j),$$

and we have the same exact decomposition on the other side, for map of covering spaces over X_F. It is not hard to believe that this construction is functorial. The following proposition establishes the main result.

Theorem 1.3.1. *The functor defined above is a category equivalence*

$$\textbf{\textit{FinÉtAlg}}/F \to \textbf{\textit{FinCov}}/X_F.$$

Proof. We are going to show that the functor is fully faithful and essentially surjective.

Let's start by showing full faithfulness. Let $Y_1 \to Y_2$ be a morphism between two connected covers of X which is a covering map itself [19, Lemma 2.2.11]. We can find a Galois cover $Y \to X$ with a cover morphism $Y \to Y_1 \to Y_2$. Both $Y \to Y_i$ are Galois covers with automorphism group corresponding to subgroups $\mathrm{Aut}(Y|Y_1) < \mathrm{Aut}(Y|Y_2) < \mathrm{Aut}(Y|X)$. In particular, every morphism $Y_1 \to Y_2$ pulls back to a morphism $Y \to Y_2$ and then lifts to an automorphism of Y; in the opposite direction, given an automorphism of Y, this induce a map between the quotients by the two subgroups. Maps between connected covers are therefore determined by the automorphisms group of a common Galois cover, hence we might reduce to this case. Consider a finite Galois extension E/F. By construction, the map $X_E \to X_F$ is the quotient map of the (proper and free) $\mathrm{Gal}(E/F)$-action; thus $\mathrm{Gal}(E/F)$ has to be the cover automorphism group, as claimed.

For essential surjectivity, observe first that

$$\varprojlim_E X_E = \varprojlim_E \mathrm{Gal}(\bar{F}/E)\backslash X_{\bar{F}} = X_{\bar{F}},$$

where the first limit is taken over finite Galois extensions E/F, and therefore the second one over a cofinal system of subgroups ordered by inclusion. The intersection of those subgroups is trivial, hence the last equality. Let $Y \to X_F$ a connected finite covering of degree d, we wish to find a finite Galois extension E/F and a morphism of covering spaces $X_E \to Y$. This would be enough to conclude: $X_E \to Y$ would be a Galois covering, hence the quotient by the subgroup $H = \mathrm{Aut}(X_E|Y) < \mathrm{Aut}(X_E|X_F)$. This correspond, by the already proven full faithfulness of our functor, to a subgroup $H < \mathrm{Gal}(E/F)$. Both X_{E^H} and Y would be quotient of X_E by the same automorphism group H, hence isomorphic.

Let's find said Galois cover. The pull-back

$$\begin{tikzcd} \tilde{Y} \arrow[r] \arrow[d] & Y \arrow[d] \\ X_{\bar{F}} \arrow[r] & X_F \end{tikzcd}$$

is trivial, because all finite covers of $X_{\bar{F}}$ are, of the same degree d; that is $\tilde{Y} = X_{\bar{F}} \times D$ for a finite set D with d elements

By a compactness argument, the pull-back must trivialize at an index E/F in the limit (the extension we are looking for): consider two complementary open subset U, V of $\tilde{Y}$ (say a connected component and its complement), each of them can be covered by a finite number of opens subset U_i, V_j that are pull-back of some open subset of some space in the projective system of the covers $(Y_E)_E$, via an argument identical to the one at the start at the proof of Lemma 1.1.5. There is a space X_{E_1} fitting in a double cartesian diagram

$$\begin{tikzcd} \tilde{Y} \arrow[r] \arrow[d] & Y_{E_1} \arrow[r] \arrow[d] & Y \arrow[d] \\ X_{\bar{F}} \arrow[r] & X_{E_1} \arrow[r] & X_F \end{tikzcd}$$

such that all U_i, V_j are pull backs of open subsets $\tilde{U}_i, \tilde{V}_j \subseteq Y_{E_1}$, where their unions

$$\bigcup_i \tilde{U}_i, \ \bigcup_j \tilde{V}_j$$

form two complementary open subsets of Y_{E_1}. That is, we found an index E_1/F such that the pull-back cover $Y_{E_1} \to X_{E_1}$ is no longer connected. We can now focus on a non-trivial connected component of Y_{E_1} and repeat the argument until, after a finite number of iterations, we get a trivial cover $Y_E \to X_E$.

The pull-back $Y_E \to X_E$ is thus isomorphic to $X_E \times D$. Let $d_0 \in D$ and consider the d_0-component:

$$\begin{tikzcd} X_E \times \{d_0\} \arrow[r, hook] \arrow[d] & X_E \times D \arrow[r, equal] \arrow[d] & Y_E \arrow[r] \arrow[d] & Y \arrow[d] \\ X_E \arrow[r, equal] & X_E \arrow[r, equal] & X_E \arrow[r] & X_F. \end{tikzcd}$$

The composition of the upper horizontal maps is a continuous map between finite covering spaces of X_F, respecting the projections to X_F, hence a covering morphism. $\square$

Corollary 1.3.2. *Let $\chi \in X_F$ a character. We have an isomorphism*

$$\mathrm{Gal}(\bar{F}/F) \simeq \pi_1^{ét}(X_F, \chi).$$

Proof. In the proof of the above theorem we showed a little more, namely that the equivalence

$$\mathbf{Fin\acute{E}tAlg}\,/F^{\mathrm{op}} \to \mathbf{FinCov}\,/X_F$$

sends the inverse system of Galois object over F isomorphically into the inverse system of Galois object over X_F. In particular

$$\begin{aligned}
\mathrm{Gal}(\bar{F}/F) &= \varprojlim\nolimits_E \mathrm{Gal}(E/F) && \text{by definition} \\
&= \varprojlim\nolimits_E \mathrm{Aut}(X_E | X_F) && \text{by full faithfulness} \\
&= \varprojlim\nolimits_E \mathrm{Aut}(Fib_\chi^{X_E}) && \text{by Galois theory} \\
&= \mathrm{Aut}(Fib_\chi) && \text{by [18, Tag 0BMU]} \\
&= \pi_1^{\text{ét}}(X_F, \chi). && \text{by definition.}
\end{aligned}$$

□

CHAPTER 2

Topological Properties

It would be nice to start this chapter with a picture of X_F and proceed to comment on its evident geometric properties but, unfortunately, it is not that simple. As we mentioned, this topological space is highly infinite-dimensional and, furthermore, we have no way to visualize the Galois action on $X_{\bar{F}}$. The only option we are left with is to compute some of its topological invariants, hoping this will shed some light on the shape of this mysterious space. We are going to describe its path components, then compute its cohomology groups.

Before we start, a warning is due to the reader. This chapter is going to get a lot more technical, straight from its first section. The number of path components of X_F does strictly depend on the nature of the inclusion $F^\times \hookrightarrow \bar{F}^\times$; thus a wide range of number theory is going to be invoked while proving the following theorems. Meanwhile, computation of the cohomology algebra is carried on in the realm of algebraic topology, and a fair familiarity with spectral sequences is needed.

2.1 Path components

Let's start by making explicit the interesting connection between the set of path components of $X_{\bar{F}}$ and the structure of the field $\bar{F}$ itself.

Consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}(\bar{F}^{\times}_{\mathrm{tf}}, \mathbb{R}) \arrow[r] \arrow[d] & \mathrm{Hom}(\bar{F}^{\times}_{\mathrm{tf}}, \mathbb{S}^1) \arrow[r, "\delta"] \arrow[d] & \mathrm{Ext}^1(\bar{F}^{\times}_{\mathrm{tf}}, \mathbb{Z}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}(\bar{F}^{\times}, \mathbb{R}) \arrow[r] \arrow[d] & \mathrm{Hom}(\bar{F}^{\times}, \mathbb{S}^1) \arrow[r, "\delta"] \arrow[d, "\alpha"] & \mathrm{Ext}^1(\bar{F}^{\times}, \mathbb{Z}) \arrow[r] \arrow[d, "\beta"] & 0 \\
 & 0 \arrow[r] & \mathrm{Hom}(\mu_\infty, \mathbb{S}^1) \arrow[r, "\delta"] \arrow[d] & \mathrm{Ext}^1(\mu_\infty, \mathbb{Z}) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

Rows are exact because part of the long exact sequences derived from

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{S}^1 \to 0. \tag{2.1}$$

The initial zeros are easy to justify: there are no non-trivial homomorphisms between $\bar{F}^{\times}$ or its torsion-free quotient and $\mathbb{Z}$, because the firsts are divisible while the latter is not, and there is no map $\mu_\infty \to \mathbb{R}$ because $\mathbb{R}$ has no torsion elements. The terminal zeros appear because $\mathbb{R}$ is divisible, hence an injective $\mathbb{Z}$-module. Columns are exact because part of the long exact sequences derived from

$$0 \to \mu_\infty \to \bar{F}^{\times} \to \bar{F}^{\times}_{\mathrm{tf}} \to 0.$$

Notice that $\mathbb{S}^1$ is divisible and $\mathrm{Ext}^2(-, \mathbb{Z})$ always vanishes, to place the remaining zeros. Commutativity follows from bifunctoriality of $\mathrm{Hom}(-,-)$.

In the first steps of our construction we fixed an element $\iota \in \mathrm{Hom}(\mu_\infty, \mathbb{S}^1)$. Its pre-image via the vertical map α is exactly $X_{\bar{F}}$ which is, by exactness, a coset of $\mathrm{Hom}(\bar{F}^{\times}_{\mathrm{tf}}, \mathbb{S}^1)$ in $\mathrm{Hom}(\bar{F}^{\times}, \mathbb{S}^1)$. We are interested in describing $\delta(\iota)$. Consider the pull-back extension

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & \mu_\infty \arrow[r] \arrow[d, hook, "\iota"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{S}^1 \arrow[r] & 0.
\end{tikzcd}$$

By the snake lemma, the central vertical map is an inclusion as well, thus M coincides with the pre-image of $\mu_\infty \subseteq \mathbb{S}^1$ in $\mathbb{R}$, that is $\mathbb{Q}$. We call this short exact sequence the "exponential extension":

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mu_\infty \to 0 \qquad \in \mathrm{Ext}(\mu_\infty, \mathbb{Z}).$$

We can apply $\mathrm{Hom}(\mu_\infty, -)$ to the above morphism of short exact sequences: in the map between the associated long exact sequences we find the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathrm{Hom}(\mu_\infty, \mu_\infty) \arrow[r, "\delta"] \arrow[d] & \mathrm{Ext}^1(\mu_\infty, \mathbb{Z}) \arrow[r] \arrow[d, equal] & 0 \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}(\mu_\infty, \mathbb{S}^1) \arrow[r, "\delta"] & \mathrm{Ext}^1(\mu_\infty, \mathbb{Z}) \arrow[r] & 0.
\end{tikzcd}$$

The left groups are zero because neither $\mathbb{Q}$ nor $\mathbb{R}$ has torsion points, while the right groups vanish because both $\mathbb{Q}$ both $\mathbb{R}$ are divisible, thus injective. The non-trivial vertical map is induced by ι and is an isomorphism by the snake lemma (or by noticing that the image of any morphism $\mu_\infty \to \mathbb{S}^1$ has to be made entirely of torsion points). In particular, $\iota \in \mathrm{Hom}(\mu_\infty, \mathbb{S}^1)$ corresponds to the identity $\mathrm{id}_{\mu_\infty} \in \mathrm{Hom}(\mu_\infty, \mu_\infty)$. The upper connecting homomorphism δ sends the identity id_{μ_∞} to the class of $\mathrm{Ext}^1(\mu_\infty, \mathbb{Z})$ corresponding to extension given by the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mu_\infty \to 0$ we started from [20, Theorem 3.4.3]. The lower connecting morphism δ in (2.1) is the same we find in the big commutative diagram (2.1); by commutativity, we can thus conclude that $\iota \in \mathrm{Hom}(\mu_\infty, \mathbb{S}^1)$ is sent by the lower isomorphism δ to the class of the exponential extension

$$[0 \to \mathbb{Z} \to \mathbb{Q} \to \mu_\infty \to 0] \in \mathrm{Ext}^1(\mu_\infty, \mathbb{Z}).$$

Back to diagram (2.1), the fiber through β over this extension is a coset of $\mathrm{Ext}^1(\bar{F}^\times_{\mathrm{tf}}, \mathbb{Z})$, that we call $\mathrm{Ext}^1_{\mathrm{exp}}(\bar{F}^\times, \mathbb{Z})$.

Proposition 2.1.1. *There is a G-equivariant bijection*

$$\pi_0(X_{\bar{F}}) \simeq \mathrm{Ext}_{\mathrm{exp}}(\bar{F}^\times, \mathbb{Z}).$$

Proof. By Proposition 1.1.2, the first row of the big diagram above

$$0 \to \mathrm{Hom}(\bar{F}^\times_{\mathrm{tf}}, \mathbb{R}) \to \mathrm{Hom}(\bar{F}^\times_{\mathrm{tf}}, \mathbb{S}^1) \to \mathrm{Ext}^1(\bar{F}^\times_{\mathrm{tf}}, \mathbb{Z}) \to 0,$$

is exactly the quotient of the middle group by the identity's path component. By commutativity of diagram, since $X_{\bar{F}}$ is a coset of $\mathrm{Hom}(\bar{F}^\times_{\mathrm{tf}}, \mathbb{S}^1) \subseteq \mathrm{Hom}(\bar{F}^\times, \mathbb{S}^1)$, its path components correspond to the classes in $\delta(X_{\bar{F}})$. In particular, since $X_{\bar{F}}$ is the pre-image of ι through the vertical map α, then $\delta(X_{\bar{F}})$ has to be the pre-image of the exponential extension through the right vertical map β, by commutativity of (2.1). $\square$

It would be interesting to understand the Galois action on $\mathrm{Ext}^1_{\mathrm{exp}}(\bar{F}^\times, \mathbb{S}^1)$, since the subset of fixed extensions $\mathrm{Ext}^1_{\mathrm{exp}}(\bar{F}^\times, \mathbb{S}^1)^G$ corresponds to the set of path components of X_F, by the previous proposition. Notice the first set has quite a lot of elements since $X_{\bar{F}}$ has an uncountable number of path-components because the solenoid does. Intuitively, if $\mathrm{Gal}(\bar{F}/F)$ is small, in some suitable sense, then it's hard to believe that it acts transitively on connected components. In this case, we expect X_F not to be path-connected.

From the above proof, we can build ourselves a better intuition of the problem. Let's focus on the first row in the big diagram (2.1) above, that is, on the identity's component. Fix a basis $\{x_i\}$ for $\bar{F}^\times_{\mathrm{tf}}$ and a section $\bar{F}^\times_{\mathrm{tf}} \to \bar{F}^\times$. A character $\chi\colon \bar{F}^\times \to \mathbb{S}^1$ is determined by the image $\chi(x_i)$ of every element in our basis and the choice of $\chi(\frac{1}{n}x_i)$ between a finite set of values, for every integer n. Formally, this is the interpretation of the short exact sequence

$$0 \to \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{S}^1) \to \mathrm{Hom}(\mathbb{Q}, \mathbb{S}^1) \to \mathrm{Hom}(\mathbb{Z}, \mathbb{S}^1) \to 0.$$

A path corresponds to a smooth change of $\chi(x_i)$, which moves the image of all n-th roots accordingly on the circle. Hence, connected components are determined by elements of the first group of the above sequence, that is, the choice of images on the inverse system of the roots $\frac{1}{n}x_i$. For the G action to identify two path-components, we thus need to have a Galois automorphism $\bar{F} \to \bar{F}$ which for every n sends $\frac{1}{n}x_i \to \zeta_n \frac{1}{n}x_i$, where ζ_n is a root of unity that depends on the components.

This interpretation is far from being precise but should give a good enough intuition to believe the following proposition, which treats the case in which there's no such automorphism. This is certain to happen when, however we choose the basis $\{x_i\}$ for $F^\times_{\mathrm{tf}}$, too many of the roots $\{\frac{1}{n}x_i\}$ are fixed by the G-action, that is, live in the base field F.

Theorem 2.1.2. *If there is an element $\alpha \in F^\times$ which is not a root of unity and has an n-th root in $F^\times$ for infinitely many integers n, then $\pi_0(X_F)$ is uncountable.*

We recall that a subgroup $B \subseteq A$ is said to be saturated if, for all elements $a \in A$ and positive integers n such that $a^n \in B$, the element a belongs to B. The saturation of a subgroup $C \subseteq A$ is the smallest saturated subgroup $B \subseteq A$ containing C.

Proof. Let M be the saturation in $F^\times$ of the infinite cyclic subgroup generated by α; notice that $\mu_\infty \subseteq M$. By hypothesis $M_{\mathrm{tf}} = M/\mu_\infty$ is a G-invariant $\mathbb{Z}$-module contained in a rank 1 $\mathbb{Q}$-vector space, which contains the free group N generated by α and $\frac{1}{n}N$ for infinitely many n. The inclusion $M \hookrightarrow \bar{F}^\times$ is dual to a surjective continuous map $X_{\bar{F}} \to \mathrm{Hom}_{\exp}(M, \mathbb{S}^1)$ (where by $\mathrm{Hom}_{\exp}$ we denote the characters that restricted to μ_∞ coincide with ι), which is G-equivariant. The target space is naturally homeomorphic to the Pontryagin dual of M_{tf}, we thus have a surjection

$$X_F \to M_{\mathrm{tf}}^\vee,$$

that is surjective on path components $\pi_0(X_F) \to \pi_0(M_{\mathrm{tf}})$. By Proposition 1.1.2, we are left to prove that $\mathrm{Ext}^1(M_{\mathrm{tf}}, \mathbb{Z})$ is uncountable.

We can show $\mathrm{Ext}^1(M_{\mathrm{tf}}, \mathbb{Z})$ is uncountable computing explicitly the cohomology group by mean of the injective resolution $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ of $\mathbb{Z}$. The group is the cokernel of the map

$$\mathrm{Hom}(M_{\mathrm{tf}}, \mathbb{Q}) \to \mathrm{Hom}(M_{\mathrm{tf}}, \mathbb{Q}/\mathbb{Z}).$$

The first group is countable, as the image of every non-zero element $m \in M_{\mathrm{tf}}$ determines a unique homomorphism. The second is uncountable: writing $M_{\mathrm{tf}} = \varinjlim_n \frac{1}{n}N$ (notice that n varies on an infinite index set $J \subseteq \mathbb{Z}$, determined by the particular structure of the module), we obtain a limit realized by

$$\{(a_n)_n \in \prod_J \mathbb{Q}/\mathbb{Z} \mid a_n = m \cdot a_{mn} \ \forall n, mn \in J\}.$$

Which is easy to see it is not countable. $\square$

Going in the opposite direction, we could ask for $F_{\mathrm{tf}}^{\times}$ to be a lattice in $\bar{F}_{\mathrm{tf}}^{\times}$, so that we would have all the desired freedom on roots of unity.

Definition 2.1.3. We say that F is a multiplicatively free field if the torsion free quotient of its multiplicative group F_{tf} is a free $\mathbb{Z}$-module.

Although this removes the obstruction of Theorem 2.1.2, the existence of those field automorphisms is not guaranteed. We are going to build the desired maps in the only conceivable way: via a limit procedure, working our way up through finite extensions of F. A slightly stronger version of the previous definition is thus required.

Definition 2.1.4. We say that F is stably multiplicatively free if all its finite extensions are multiplicatively free.

Fields satisfying this hypothesis (and the one on roots of unity) exist and are of some interest. We can fix a significant example.

Theorem 2.1.5. *The field $\mathbb{Q}(\mu_\infty)$ is stably multiplicatively free.*

Proof. By the Kronecker–Weber theorem we know $\mathbb{Q}(\mu_\infty)$ to be the maximal abelian extension of $\mathbb{Q}$. The following argument is divided in two main steps: first, we prove that all finite extensions of $\mathbb{Q}$ are multiplicatively free, then show how this implies the claim for all finite extensions $F/\mathbb{Q}(\mu_\infty)$.

Let $K/\mathbb{Q}$ be a number field and $\mathcal{O}_K$ be the ring of integers of K, the proof relies on two main arithmetic facts:

1. The Dirichlet's Unit Theorem, stating that the group $\mathcal{O}_K^\times$ of integral units is isomorphic to a product
$$\mathcal{O}_K^\times = T \times \mathbb{Z}^r,$$
where T is a finite group and the rank r is a positive integer.

2. The group J of fractional ideals of $\mathcal{O}_K$ is freely generated by prime ideals [11, Chapter 3, Theorem 16, ex. 31 (c)].

One has a surjective map from $K^\times$ to the subgroup $P \subseteq J$ of principal ideals, sending every element $x \in K^\times$ to the ideal it generates. The kernel of this map is the group of units $\mathcal{O}_K^\times$. Hence, we have a short exact sequence

$$0 \to \mathcal{O}_K^\times \to K^\times \to P \to 0.$$

Notice that $P \subseteq I$ is a submodule of a free module (by 2), hence a free module itself. Therefore the exact sequence splits, producing an isomorphism

$$K^\times = P \times \mathcal{O}_K^\times = P \times \mathbb{Z}^r \times T,$$

where we used (1) as well. When we pass to the torsion free quotient, we kill T and remain with a free group, hence the claim.

Let's extend the result to finite extensions $F/\mathbb{Q}(\mu_\infty)$: we are going to prove that every finitely generated subgroup $A \subseteq F_{\mathrm{tf}}^\times$ has a finitely generated saturation A_{sat}^F, so that we can apply [12, Lemma] to the countable group $F_{\mathrm{tf}}^\times$ to conclude it is free.

Since $\mathbb{Q}(\mu_\infty)/\mathbb{Q}$ is an abelian extension, we can find a finite extension $K'/\mathbb{Q}$ such that F/K' is abelian: in order to do that, notice that F is a simple extension $\mathbb{Q}(\mu_\infty)$, let's say $F = \mathbb{Q}(\mu_\infty, \alpha)$ for some $\alpha \in F$; then $K' = \mathbb{Q}(\alpha)$ is finite over $\mathbb{Q}$ and F/K' is Galois with a group isomorphic to a subgroup of $\mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q})$, hence abelian.

Let K be the finite extension of K' generated by the finitely many generators of A (a lift of the generators in F) and i (so that we don't have to worry about a special case later), so that $A \subseteq K_{\mathrm{tf}}^\times \subseteq F_{\mathrm{tf}}^\times$ and $K/\mathbb{Q}$ is finite.

Since $K_{\mathrm{tf}}^\times$ is free (K being a finite extension of $\mathbb{Q}$), the saturation A_{sat}^K of A in $K_{\mathrm{tf}}^\times$ is free and finitely generated. Consider the inclusion

$$0 \to A_{\mathrm{sat}}^K \to A_{\mathrm{sat}}^F \to Q \to 0.$$

Notice that upon taking $- \otimes \mathbb{Q}$ all saturations are sent to $A_{\mathrm{sat}}^{\bar{F}}$, the saturation in the algebraic closure, hence the quotient Q is a torsion group. We wish to show that Q is finite, so that A_{sat}^F has to be generated by finitely many elements as well: the union of those generating A_{sat}^K and a lift of those generating the quotient Q. We are going to prove that, for any prime p, the following holds:

- If $\zeta_p \in K$, then the p-torsion $Q[p]$ is finite.
- If $\zeta_p \notin K$, then the p-torsion $Q[p] = 0$ is trivial.

The claim follows since K contains finitely many roots of unity.

Let n be the positive integer such that $\zeta_{p^n} \notin K$ while $\zeta_{p^{n-1}} \in K$. We claim that $Q[p]$ has exponent p^{n-1} (that is, multiplication by p^{n-1} kills the whole quotient). For sake of contradiction, suppose the converse: let $a \in F$ be an element whose image in Q has order p^n and $b = a^{p^n} \in K$.

First we show that $\mu(x) = x^{p^n} - b$ is the minimal polynomial of a over K. We prove it is irreducible: suppose not, then by [6, Theorem 8.1.6] there exists $b' \in K$ such that $(b')^p = b$, thus μ has a factor of the form $x^{p^{n-1}} - b'$. There is a root of μ, let's say $a\zeta_{p^n}^k$, such that

$$\left(a\zeta_{p^n}^k\right)^{p^{n-1}} = b' \in K.$$

In particular a has order p^{n-1} in Q, providing a contradiction: μ is irreducible, hence the minimal polynomial of a over K.

Notice that, because F/K is abelian, its intermediate extension $K(a)/K$ has to be normal: $\mu(x)$ splits in $K(a)$. In $K(a)$ we find every root of $\mu(x)$ and

$$\zeta_{p^n} = \frac{\zeta_{p^n} a}{a}, \qquad \text{that is } K(\zeta_{p^n}) \subseteq K(a).$$

Since $K \neq K(\zeta_{p^n})$, $x^{p^n} - b$ is reducible in $K(\zeta_{p^n})$; arguing as above, we find that $a^{p^{n-1}} \in K(\zeta_{p^n})$, hence

$$K \subsetneq K(a^{p^{n-1}}) \subseteq K(\zeta_{p^n}) \subseteq K(a).$$

- If $\zeta_p \notin K$, which is to say $n = 1$, the above chain of inclusions forces the equality $K(a) = K(\zeta_p)$. These extensions have different degrees over K, providing a contradiction.
- If $\zeta_p \in K$, which is to say $n > 1$, then $K(a^{p^{n-1}}) = K(\zeta_{p^n})$, since both extensions have degree p over K. Since $\zeta_p \in K$, Kummer theory tells us that degree p cyclic extensions are in one-to-one correspondence with elements of $K^\times/K^{\times p}$. In particular our extension is obtained adding p-th roots of elements in a single coset of $K^{\times p}$, which means we can find $u \in K^\times$ such that $a^{p^{n-1}} = u\zeta_{p^n}$, which implies the desired contradiction: $a^{p^{n-1}}$ is trivial in Q.

□

For this type of field, the Galois group is "big "enough to act transitively on path components, hence the following result.

Theorem 2.1.6. *If F is countable and stably multiplicatively free, then X_F is path connected.*

The proof is pretty involved, we thus start by giving a rough outline. The starting observation is that, the field being countable, we can choose a filtration

$$V_0 \subseteq V_1 \subseteq V_2 \subseteq \dots$$

of $\bar{F}^\times$, where every V_n is a G-stable saturated subgroup such that $(V_n)_{\mathrm{tf}} = V_n/\mu_\infty$ is a vector space over $\mathbb{Q}$ of finite rank.

Using the exponential extension $0 \to \mathbb{Z} \to \mathbb{Q} \to \mu_\infty \to 0$ as an injective resolution of $\mathbb{Z}$, we get an inverse system of defining short exacts sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(V_{n+1}, \mathbb{Q}) \arrow[r] \arrow[d] & \mathrm{Hom}(V_{n+1}, \mu_\infty) \arrow[r] \arrow[d] & \mathrm{Ext}^1(V_{n+1}, \mathbb{Z}) \arrow[r] \arrow[d, dashed] & 0 \\ 0 \arrow[r] & \mathrm{Hom}(V_n, \mathbb{Q}) \arrow[r] & \mathrm{Hom}(V_n, \mu_\infty) \arrow[r] & \mathrm{Ext}^1(V_n, \mathbb{Z}) \arrow[r] & 0. \end{tikzcd}$$

Where the first two vertical maps come from the inclusion $V_n \subseteq V_{n+1}$ and are easily seen to be surjective, while the dashed one exists by the universal property of the cokernel and is surjective by the Snake Lemma. Taking the limit we get

$$0 \to \mathrm{Hom}(\bar{F}^\times, \mathbb{Q}) \to \mathrm{Hom}(\bar{F}^\times, \mu_\infty) \to \varprojlim_n \mathrm{Ext}^1(V_n, \mathbb{Z}) \to 0,$$

which, since the limit functor is left exact and all transition maps between Exts are surjective, is exact [18, Tag 02N1]. The short exact sequence we got is the defining exact sequence of $\mathrm{Ext}^1(\bar{F}^\times, \mathbb{Z})$: the natural map

$$\mathrm{Ext}^1(\bar{F}^\times, \mathbb{Z}) \to \varprojlim_n \mathrm{Ext}^1(V_n, \mathbb{Z})$$

is an isomorphism. Call $\mathrm{Hom}_{\mathrm{exp}}(V, \mu_\infty)$ the subset of $\mathrm{Hom}(V, \mu_\infty)$ of characters that restrict to the identity on roots of unity and $\mathrm{Ext}^1_{\mathrm{exp}}(V, \mu_\infty)$ its image via the connecting morphism of the long exact sequence associated to the exponential extension. The above isomorphism restricts to a bijection

$$\mathrm{Ext}^1_{\mathrm{exp}}(\bar{F}^\times, \mathbb{Z}) \to \varprojlim_n \mathrm{Ext}^1_{\mathrm{exp}}(V_n, \mathbb{Z}). \tag{2.2}$$

Here the proof separates into two distinct natural steps. First, we are going to solve the finitely generated case, by showing that G acts transitively on $\mathrm{Ext}^1_{\mathrm{exp}}(V_n, \mathbb{Z})$. Afterward, we will show how this implies that the whole G-action on $\mathrm{Ext}^1_{\mathrm{exp}}(\bar{F}^\times, \mathbb{Z})$ is transitive, by a limit argument that exploits (2.2).

Lemma 2.1.7. *Let $V \subseteq \bar{F}^\times$ be a G-stable saturated subgroup such that V_{tf} is a finite rank vector space. The induced G action on $\mathrm{Ext}^1_{exp}(V, \mathbb{Z})$ is transitive.*

Proof. Fix a basis of V_{tf} and lift it to a finite subset $B \subset \bar{F}^{\times}$. Let $E = F(B)$ be the finite extension of F generated by B, which, by hypothesis, is multiplicatively free. Let $\Lambda = V \cap E^{\times}$; this is a subgroup of V containing μ_∞, and the quotient $\Lambda_{\mathrm{tf}} = \Lambda/\mu_\infty$ is a full $\mathbb{Z}$-lattice in V_{tf}. Notice that V is the saturation of Λ in $\bar{F}^{\times}$.

The first step of this proof is arithmetic. We need to understand the G-action on $\mathrm{Hom}(V, \mu_\infty)$ and, in order to do so, we are going to describe explicitly the action of the subgroup $\mathrm{Gal}(\bar{F}/E)$. Notice that E contains all roots of unity hence, by Kummer theory [8, Chapter VI, Theorem 8.1], the extension $E(V)/E$ we get adding all elements of V is abelian, since we are adding roots of elements in E only. It is then easy to check that the biadditive pairing

$$\mathrm{Gal}(E(V)/E) \times V/\Lambda \to \mu_\infty, \qquad (\sigma, x) \mapsto \langle \sigma, x \rangle = \frac{\sigma(x)}{x},$$

known as the Kummer pairing, is non-degenerate. In particular, this pairing induces a duality isomorphism [8, Chapter I, Theorem 9.2]

$$\kappa\colon \mathrm{Gal}(E(V)/E) \to \mathrm{Hom}(V/\Lambda, \mu_\infty), \qquad \sigma \mapsto \langle \sigma, - \rangle.$$

This means that we can identify $\mathrm{Gal}(E(V)/E)$ with the subgroup $\mathrm{Hom}(V/\Lambda, \mu_\infty) \subseteq \mathrm{Hom}(V, \mu_\infty)$, that is the subgroup of characters $\chi\colon V \to \mu_\infty$ that are trivial on Λ. Exploiting this isomorphism we can explicitly describe the natural $\mathrm{Gal}(E(V)/E)$-action on $\mathrm{Hom}(V, \mu_\infty)$, that is the one induced by the natural G-action on V, which can be restricted to the coset $\mathrm{Hom}_{\exp}(V, \mu_\infty)$ of characters which are the identity on μ_∞, because $\mu_\infty \subseteq F \subseteq E$. Since any character $\chi \in \mathrm{Hom}_{\exp}(V, \mu_\infty)$ fixes roots of unity, the Kummer map allows for a crucial simplification

$$\chi(\sigma^{-1}(\alpha)) = \chi\left(\frac{\sigma^{-1}(\alpha)}{\alpha}\right) \cdot \chi(\alpha) = \frac{\sigma^{-1}(\alpha)}{\alpha} \cdot \chi(\alpha),$$

thus

$$\sigma(\chi) = \kappa(\sigma)^{-1} \cdot \chi.$$

That is, through our identification κ, the Galois action of $\mathrm{Gal}(E(V)/E)$ coincides with the action of $\mathrm{Hom}(V/\Lambda, \mu_\infty)$ on $\mathrm{Hom}_{\exp}(V, \mu_\infty)$ by multiplication (where we intend everything is happening inside the ambient space $\mathrm{Hom}(V, \mu_\infty)$). In particular, for every character $\chi \in \mathrm{Hom}_{\exp}(V, \mu_\infty)$, the subspace $\mathrm{Hom}(V/\Lambda, \mu_\infty) \cdot \chi$ of $\mathrm{Hom}_{\exp}(V, \mu_\infty)$ is a $\mathrm{Gal}(E(V)/E)$-orbit. Or, to state this in ever simpler terms, G permutes transitively the elements of every subspace of the form $\mathrm{Hom}(V/\Lambda, \mu_\infty) \cdot \chi$.

The second step of this proof requires some topological considerations. Consider the following big commutative diagram, induced by $0 \to \mu_\infty \to V \to V_{\mathrm{tf}} \to 0$ and the exponential short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mu_\infty \to 0$:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(V_{\mathrm{tf}}, \mathbb{Q}) & \longrightarrow & \mathrm{Hom}(V_{\mathrm{tf}}, \mu_\infty) & \longrightarrow & \mathrm{Ext}^1(V_{\mathrm{tf}}, \mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(V, \mathbb{Q}) & \longrightarrow & \mathrm{Hom}(V, \mu_\infty) & \xrightarrow{\delta} & \mathrm{Ext}^1(V, \mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathrm{Hom}(\mu_\infty, \mu_\infty) & \longrightarrow & \mathrm{Ext}^1(\mu_\infty, \mathbb{Z}) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}
$$

Exactness and commutativity follow from the same arguments we used for the previous big diagram (2.1). Recall that $\mathrm{Ext}^1_{\mathrm{exp}}(V, \mathbb{Z})$ is the image via the quotient map δ of $\mathrm{Hom}_{\mathrm{exp}}(V, \mu_\infty)$, on which we explicitly know the $\mathrm{Gal}(\bar{F}/E)$-action. We are going to prove that:

- **The action is transitive on an open subspace of $Y \subseteq \mathrm{Hom}_{\mathrm{exp}}(V, \mu_\infty)$.** Consider the subgroup $\mathrm{Hom}(V/\Lambda, \mu_\infty) \subseteq \mathrm{Hom}(V_{\mathrm{tf}}, \mu_\infty)$. This is an open subgroup. In order to see this, we write $V_{\mathrm{tf}} = \Lambda_{\mathrm{tf}} \otimes \mathbb{Q}$ to get the isomorphism

$$
\begin{aligned}
\mathrm{Hom}(V_{\mathrm{tf}}, \mu_\infty) &= \mathrm{Hom}(\Lambda_{\mathrm{tf}}, \mathrm{Hom}(\mathbb{Q}, \mu_\infty)) \\
\varphi &\mapsto [\tilde{\varphi} \colon \lambda \mapsto \varphi(\lambda \cdot -)]
\end{aligned}
$$

  Notice this is an isomorphism of topological groups, where all Homs are intended with the compact-open topology and both Λ_{tf} and $\mathbb{Q}$ are discrete, while μ_∞ has the euclidean topology. For any $x \in V_{\mathrm{tf}}$ there is an integer m such that $m \cdot x \in \Lambda_{\mathrm{tf}} \subseteq V_{\mathrm{tf}}$. The open subset of the left hand side space $V_1(\{x\}, U)$ of characters $\varphi \colon V_{\mathrm{tf}} \to \mu_\infty$ such that $\varphi(x)$ belongs to the open subset $U \subseteq \mu_\infty$, corresponds to the open subset of the right hand space $V(\{(m \cdot x, 1/m)\}, U)$ of bilinear maps $\tilde{\varphi} \colon \Lambda_{\mathrm{tf}} \times \mathbb{Q} \to \mu_\infty$ such that $\tilde{\varphi}(m \cdot x, 1/m)$ belongs to U. A compact discrete space is finite, hence the compact-open topology is generated in both spaces by sets of this form, letting x and U vary, thus the claim.

  We can think at $\mathrm{Hom}(\mathbb{Q}, \mu_\infty)$ as the subspace of the solenoid $\mathcal{S} = \mathrm{Hom}(\mathbb{Q}, \mathbb{S}^1)$ of those characters $\mathbb{Q} \to \mathbb{S}^1$ whose image is contained in $\iota \colon \mu_\infty \hookrightarrow \mathbb{S}^1$. This corresponds to the image of the subgroup $\mathbb{A}^{\mathrm{fin}}_{\mathbb{Q}}$ of finite adeles in the solenoid $\mathcal{S} = \mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$, by Lemma 1.1.6. Through this identification, the subset $\mathrm{Hom}(V/\Lambda, \mu_\infty) \subseteq \mathrm{Hom}(V_{\mathrm{tf}}, \mu_\infty)$ is the one corresponding to the inclusion

$$\hat{\mathbb{Z}} = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mu_\infty) \hookrightarrow \mathrm{Hom}(\mathbb{Q}, \mu_\infty) = \mathbb{A}^{\mathrm{fin}}_{\mathbb{Q}}.$$

Therefore $\mathrm{Hom}(\Lambda_{\mathrm{tf}}, \hat{\mathbb{Z}})$ is open in $\mathrm{Hom}(\Lambda_{\mathrm{tf}}, \mathbb{A}_{\mathbb{Q}}^{\mathrm{fin}})$. Let $\chi \in \mathrm{Hom}_{\mathrm{exp}}(V, \mu_\infty)$ and Y the coset $\mathrm{Hom}(V/\Lambda, \mu_\infty) \cdot \chi$; this coset is open and, by the previous part of the proof, the G action is transitive on Y.

- **Every fiber of the projection** $\mathrm{Hom}(V, \mu_\infty) \to \mathrm{Ext}^1(V, \mathbb{Z})$ **is a dense subspace of the domain.** Since the first row tells us that fibers are cosets of $\mathrm{Hom}(V_{\mathrm{tf}}, \mathbb{Q})$, it is enough to prove this subgroup is dense in $\mathrm{Hom}(V_{\mathrm{tf}}, \mu_\infty)$. Under the identification above and Lemma 1.1.6, this inclusion corresponds to the map
$$\mathbb{Q} = \mathrm{Hom}(\mathbb{Q}, \mathbb{Q}) \hookrightarrow \mathrm{Hom}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) = \mathbb{A}_{\mathbb{Q}}^{\mathrm{fin}},$$
hence the claim follows from the strong approximation theorem [1, Chapter II, 15].

From these observations, the claim follows. Each fiber of the projection

$$\delta\colon \mathrm{Hom}_{\mathrm{exp}}(V, \mu_\infty) \to \mathrm{Ext}^1_{\mathrm{exp}}(V, \mathbb{Z})$$

is dense, hence intersects the open subspace $Y \subseteq \mathrm{Hom}(V, \mu_\infty)$. That is, every element of the quotient $\mathrm{Ext}^1_{\mathrm{exp}}(V, \mathbb{Z})$ has a representative in Y. But we know any two such representatives are conjugated by an element of G. Hence the G action is transitive on the quotient, as claimed. □

Let's get back to the proof of our theorem. We need a limit argument to show that G acts transitively on connected components. Notice that the exponential short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mu_\infty \to 0$ induces surjective maps

$$\delta\colon \mathrm{Hom}(V_n, \mu_\infty) \to \mathrm{Ext}^1(V_n, \mathbb{Z})$$

for every integral n, all connected by surjective restriction morphisms. These induce a surjective map $\varprojlim_n \mathrm{Hom}(V_n, \mu_\infty) \to \varprojlim_n \mathrm{Ext}^1(V_n, \mathbb{Z})$.

Let $\epsilon_1, \epsilon_2 \in \mathrm{Ext}^1_{\mathrm{exp}}(\bar{F}^\times, \mathbb{Z})$ be two path-components and consider their restriction

$$\epsilon_1^{(n)}, \epsilon_2^{(n)} \in \mathrm{Ext}^1_{\mathrm{exp}}(V_n, \mathbb{Z}).$$

We know, by the just proven Lemma 2.1.7, that their restrictions are conjugated by G. We rigidify the situation in the following way: there exist characters

$$\chi_1^{(n)}, \chi_2^{(n)} \in \mathrm{Hom}(V_n, \mu_\infty)$$

in the same G-orbit, mapped by δ on the respective $\epsilon_i^{(n)}$. We wish to show that we can find a sequence of such $\chi_i^{(n)}$ compatible with the restriction maps!

Lemma 2.1.8. *There exist characters* $\chi_1^{(n+1)}, \chi_2^{(n+1)} \in \mathrm{Hom}_{\exp}(V_{n+1}, \mu_\infty)$ *in the same* G *orbit such that both* $\delta\colon \chi_i^{(n+1)} \mapsto \epsilon_i$ *and* ${\chi_i^{(n+1)}}_{|V_n} = \chi_i^{(n)}$.

Proof. All characters of the proposition live in the big commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(V_{n+1}/V_n, \mathbb{Q}) & \longrightarrow & \mathrm{Hom}(V_{n+1}/V_n, \mu_\infty) & \longrightarrow & \mathrm{Ext}^1(V_{n+1}/V_n, \mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(V_{n+1}, \mathbb{Q}) & \longrightarrow & \mathrm{Hom}(V_{n+1}, \mu_\infty) & \xrightarrow{\delta} & \mathrm{Ext}^1(V_{n+1}, \mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(V_n, \mathbb{Q}) & \longrightarrow & \mathrm{Hom}(V_n, \mu_\infty) & \xrightarrow{\delta} & \mathrm{Ext}^1(V_n, \mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Since $\mathrm{Hom}(V_{n+1}/V_n, \mu_\infty) \subseteq \mathrm{Hom}(V_{n+1}, \mu_\infty)$ is open, while $\mathrm{Hom}(V_{n+1}, \mathbb{Q}) \subseteq \mathrm{Hom}(V_{n+1}, \mu_\infty)$ is dense, so are their cosets. In particular, the fiber of $\chi_1^{(n)}$ via the vertical map is open, the fiber of ϵ_1 via δ is dense, hence they intersect in an element we call $\chi_1^{(n+1)}$. By construction, $\chi_1^{(n+1)}$ has all the desired properties.

We choose an element $\sigma \in G$ such that $\sigma\left(\chi_1^{(n)}\right) = \chi_2^{(n)}$. Consider $\sigma\left(\chi_1^{(n+1)}\right)$. This character projects onto $\chi_2^{(n)}$, but might not be in the correct path component, namely

$$\epsilon_2 - \sigma\left(\chi_1^{(n+1)}\right) \quad \in \mathrm{Ext}^1(V_{n+1}, \mathbb{Z})$$

could not be zero. Although, by construction, this difference lives in the kernel of the vertical restriction map, that is $\mathrm{Ext}^1(V_{n+1}/V_n, \mathbb{Z})$. By diagram chasing, we find an element $\alpha \in \mathrm{Hom}(V_{n+1}/V_n, \mu_\infty) \subseteq \mathrm{Hom}(V_{n+1}, \mu_\infty)$, such that

$$\delta(\alpha) = \epsilon_2 - \sigma\left(\chi_1^{(n+1)}\right).$$

We define $\chi_2^{(n+1)} = \sigma\left(\chi_1^{(n+1)}\right) + \alpha$. By construction, we now have

$$\chi_2^{(n+1)}|_n = \sigma\left(\chi_1^{(n+1)}|_n\right) + 0 = \chi_1^{(n)},$$
$$\delta\left(\chi_2^{(n+1)}\right) = \sigma\delta\left(\chi_1^{(n+1)}\right) + \delta(\alpha) = \epsilon_2.$$

From the discussion in the previous proof, we know there is a surjective homomorphism from an open subgroup $H < G$ to

$$H \to \mathrm{Hom}(V_{n+1}/\Lambda, \mu_\infty) \to \mathrm{Hom}(V_{n+1}/V_n, \mu_\infty);$$

that is, there exists an element $\tau \in G$ such that $\kappa(\tau) = \alpha$. Then

$$\tau\sigma\left(\chi_1^{(n+1)}\right) = \sigma\left(\chi_1^{(n+1)}\right) + \kappa(\tau) = \chi_2^{(n+1)}.$$

□

This lemma allows us to build two sequences $\chi_1, \chi_2 \in \varprojlim_n \mathrm{Hom}(V_n, \mu_\infty)$ which are mapped by the connecting morphism δ to the connected components ϵ_1, ϵ_2. We built these characters such that the subsets

$$T_n = \{\sigma \in G \mid \sigma\left(\chi_1^{(n)}\right) = \chi_2^{(n)}\}$$

are not empty. They are closed and contained in one another, forming a nested chain of compact subspaces of G. In particular, their intersection is non-empty too. That is what we wanted, since an element

$$\sigma \in \bigcap_n T_n$$

by definition sends $\chi_1 \mapsto \chi_2$, hence $\epsilon_1 \mapsto \epsilon_2$.

2.2 Cohomology groups

We proceed in the task of computing topological invariants of the space X_F. Since our space is horrendous, or -to be fair- just pathological, singular cohomology is not an interesting enough set of invariants. For example, let's consider the case of the solenoid: its cohomology groups are direct sums of the cohomology of its path-components, which are homeomorphic to $\mathbb{R}$, hence trivial. In the same way, singular cohomology misses the information about X_F we would like to retrieve. We are going to compute its sheaf cohomology instead, which has the nice property of behaving nicely under limits.

Proposition 2.2.1 ([2, Chapter X, Theorem 2.1])**.** *Let $(X_n)_n$ be a cofiltered projective system of compact Hausdorff topological spaces, let A an abelian group and the corresponding constant sheaf over the X_n. We then have a canonical isomorphism*

$$\mathrm{H}^p\left(\varprojlim_n X_n, A\right) = \varinjlim_n \mathrm{H}^p\left(X_n, A\right)$$

in sheaf cohomology.

This gives us a way to compute the cohomology of $X_{\bar{F}}$ for some selected abelian groups A. Afterwards we'll handle the descent to X_F.

- A **is a torsion abelian group.** Since A can be expressed as a combination of filtered colimits of finite groups and this operation commutes with sheaf cohomology over compact Hausdorff spaces [18, Tag 01FF], it's enough to compute the case $A = \mathbb{Z}/n\mathbb{Z}$. As usual, when we need to understand geometric properties we pass to the homeomorphic coset $\mathrm{Hom}(\bar{F}^\times_{\mathrm{tf}}, \mathbb{S}^1)$. Let's write $\bar{F}^\times_{\mathrm{tf}}$ as a colimit over the filtered system of its finitely generated free subgroups $\varinjlim V$ and apply Proposition 2.2.1 we get: $\mathrm{H}^p(X_{\bar{F}}, A) = \varinjlim \mathrm{H}^p(V^\vee, A)$. In this direct system we find, for every V, an arrow induced by the inclusion $V \to \frac{1}{n}V$, which corresponds to a map $(\frac{1}{n}V)^\vee \to V^\vee$ between the duals, which happen to be tori of the same dimension. We understand the map induced in cohomology since on such well-behaved spaces we have a canonical isomorphism between sheaf and singular cohomology! And we know the latter from [9, Corollary 1.3.2.], for example. In particular,

$$\mathrm{H}^p(V^\vee, \mathbb{Z}) = \bigwedge^p \mathrm{Hom}(V, \mathbb{Z});$$

that is, all cohomology groups are free and the cohomology algebra $\mathrm{H}^\bullet(V^\vee, \mathbb{Z})$ is generated in degree 1 by $x_1, \ldots, x_k$, where k is the rank of V. In particular, the map induced by $V \to \frac{1}{n}V$ corresponds to $x_i \mapsto n \cdot x_i$. Applying the universal coefficient theorem we get

$$\mathrm{H}^\bullet(V^\vee, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z} \otimes \mathrm{H}^\bullet(V^\vee, \mathbb{Z})$$

is generated in degree 1 by the same k generators, and our map kills everything in degree $p > 0$. We conclude that

$$\mathrm{H}^0(X_{\bar{F}}, A) = A, \qquad \mathrm{H}^p(X_{\bar{F}}, A) = 0 \quad \text{for all } p > 0.$$

- $A = \mathbb{Q}$**, rational coefficients.** This is a straightforward application of the universal coefficient theorem to the well-known cohomology algebra of the n-torus:

$$\mathrm{H}^p(V^\vee, \mathbb{Q}) = \mathrm{H}^p(V^\vee, \mathbb{Z}) \otimes \mathbb{Q} = \bigwedge^p_{\mathbb{Z}} \mathrm{H}^1(V^\vee, \mathbb{Z}) \otimes \mathbb{Q} = \bigwedge^p_{\mathbb{Z}} V \otimes \mathbb{Q}.$$

Since exterior product commutes with our colimit and $\bar{F}^\times_{\mathrm{tf}}$ is a $\mathbb{Q}$-vector space, we get

$$\mathrm{H}^p(X_{\bar{F}}, \mathbb{Q}) = \bigwedge^p_{\mathbb{Q}} \bar{F}^\times_{\mathrm{tf}} \quad \text{for all } p.$$

- $A = \mathbb{Z}$**, integral coefficients.** To compute these groups we have two possible approaches. One, is applying the same argument; that is, we know the cohomology of tori and the exterior product commute with the direct limit, and that gives directly

$$\varinjlim \mathrm{H}^p(V^\vee, \mathbb{Z}) = \varinjlim \bigwedge_{\mathbb{Z}}^{p} \mathrm{H}^1(V^\vee, \mathbb{Z}) = \bigwedge_{\mathbb{Z}}^{p} \varinjlim V = \bigwedge_{\mathbb{Q}}^{p} \bar{F}_{\mathrm{tf}}^\times.$$

  Another approach is possible: from the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ we get a long one of which all groups except the one with integral coefficient we already computed! It starts with

$$0 \to \mathrm{H}^0(X_{\bar{F}}, \mathbb{Z}) \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to \mathrm{H}^1(X_{\bar{F}}, \mathbb{Z}) \to \bar{F}_{\mathrm{tf}}^\times \to 0$$

  and then continues with isomorphism between the groups with integral and rational coefficients, since all others are 0, $\mathbb{Q}/\mathbb{Z}$ being a torsion group. The map $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ in the above long sequence is surjective, hence the isomorphism in degree 1. To sum up

$$\mathrm{H}^0(X_{\bar{F}}, \mathbb{Z}) = \mathbb{Z}, \qquad \mathrm{H}^p(X_{\bar{F}}, \mathbb{Z}) = \mathrm{H}^p(X_{\bar{F}}, \mathbb{Q}) = \bigwedge_{\mathbb{Q}}^{p} \bar{F}_{\mathrm{tf}}^\times \quad \text{for all } p > 0.$$

We now handle the descent to X_F. Let's put ourselves in the very general setting of a compact Hausdorff space X, a profinite group G operating continuously and freely on X and the quotient space $Y = G\backslash X$. Any G-equivariant sheaf on G can be pushed through the quotient map to a sheaf on Y. Connecting the cohomologies of this spaces there is a spectral sequence, the so called Cartan-Leray spectral sequence (look at [13, p. 337])

$$E_2^{p,q} = \mathrm{H}^p(G, \mathrm{H}^q(X, A)) \Rightarrow \mathrm{H}^{p+q}(Y, A),$$

where $\mathrm{H}^p(G, \bullet)$ denotes continuous group cohomology, and $\mathrm{H}^q(X, A)$ and $\mathrm{H}^{p+q}(X, A)$ denote sheaf cohomology. Letting $X = X_{\bar{F}}$, G our absolute Galois group and $Y = X_F$ we retrieve

$$E_2^{p,q} = \mathrm{H}^p(G, \mathrm{H}^q(X_{\bar{F}}, A)) \Rightarrow \mathrm{H}^{p+q}(X_F, A).$$

This allows us to compute the cohomology groups of X_F:

- A **is a torsion abelian group.** Let's take a look at the second page of the Cartan-Leray spectral sequence:

$$E_2^{p,q} = \mathrm{H}^p(G, \mathrm{H}^q(X_{\bar{F}}, A)).$$

From our previous work, we know this to be zero for $q > 0$. Thus the sequence collapses at the second page leaving us on the first row the groups we were looking for

$$\mathrm{H}^p(X_F, A) = \mathrm{H}^p(G, A). \tag{2.3}$$

- $A = \mathbb{Q}$**, rational coefficients.** All the cohomology groups $\mathrm{H}^q(X_{\bar{F}}, \mathbb{Q})$ are $\mathbb{Q}$-vector spaces, hence have trivial Galois cohomology, so that the spectral sequence collapses again at the second page and we obtain isomorphisms

$$\mathrm{H}^q(X_F, \mathbb{Q}) = \mathrm{H}^0(G, \textstyle\bigwedge_{\mathbb{Q}}^q \bar{F}_{\mathrm{tf}}^{\times}) = \left(\textstyle\bigwedge_{\mathbb{Q}}^q \bar{F}_{\mathrm{tf}}^{\times}\right)^G. \tag{2.4}$$

- $A = \mathbb{Z}$**, integral coefficients.** In this case, the cohomology groups do not have any easy presentation, all we can say is what we can derive from the long exact sequence derived from $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. Since the last map in degree zero in surjective, the first connecting homomorphism is zero and in this sequence we find

$$0 \to \mathrm{H}^1(X_F, \mathbb{Z}) \to \left(\bar{F}_{\mathrm{tf}}^{\times}\right)^G \to \mathrm{H}^1(G, \mathbb{Q}/\mathbb{Z}) \to \dots$$

A bit more can be said about cohomology groups with integral coefficients, but to do so, one has to enter the realm of K-theory. To be fair, we are only going to need one theorem. We refer to [4] for an introduction to the subject, while we collect the main facts we are going to use.

Let's recall that the n-th Milnor group $K_n^M(F)$ is defined as the the quotient of the tensor product $(F^\times)^{\otimes n}$ by the ideal generated by $a_1 \otimes \cdots \otimes a_n$, where $a_i + a_j = 1$. From the definition follows elementary that $a_1 \otimes \cdots \otimes a_n$ is zero in $K_n^M(F)$ when two of the factors a_i, a_j are equal; that is, the group could be defined as well as a quotient of the exterior product of n-copies of $F^\times$. From Kummer Theory, we have a map

$$\partial_m \colon F^\times \to \mathrm{Hom}(G, \mu_m),$$

induced in cohomology by multiplication by m in $\bar{F}^\times$. The map

$$\partial_m^n \colon (F^\times)^{\otimes n} \to \mathrm{H}^n(G, \mu_m)^{\otimes n} \to \mathrm{H}^1(G, \mu_m^{\otimes n}),$$

which we get by tensoring ∂ with itself n times, then composing with the cup product, factors through the n-th Milnor group. That is, it defines a map known as the Galois symbol

$$\partial_m^n \colon K_n^M(F) \to \mathrm{H}^n(G, \mu_m^{\otimes n}).$$

The norm residue isomorphism theorem of Voevodsky [4, Theorem 4.6.5], also known as the Bloch-Kato conjecture, states that for any field F, for every integer $n \geq 0$ and any integer m that is prime to characteristic of F, the map

$$\partial_m^n \colon K_n^M(F) \otimes \mathbb{Z}/m\mathbb{Z} \to \mathrm{H}^n(G, \mu_m^{\otimes n})$$

is an isomorphism. Notice that, under our assumption that $\mu_m \subseteq F$, the G action on μ_m is trivial, thus the last group is simply the cohomology with coefficients in $\mu_m^{\otimes n} = \mu_m = \mathbb{Z}/m\mathbb{Z}$.

Proposition 2.2.2. *The homomorphisms* $\mathrm{H}^p(X_F, \mathbb{Q}) \to \mathrm{H}^p(X_F, \mathbb{Q}/\mathbb{Z})$*, induced by* $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$*, are all surjective.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} (\bigwedge^p F^\times \otimes \mathbb{Q}) & \longrightarrow & K_p^M(F) \otimes \mathbb{Q}/\mathbb{Z} \\ \big\downarrow & & \big\downarrow \partial \\ \mathrm{H}^p(X_F, \mathbb{Q}) & \longrightarrow & \mathrm{H}^p(X_F, \mathbb{Q}/\mathbb{Z}). \end{array}$$

The bottom arrow is induced by the projection $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$. The right one is obtained by taking a colimit over the isomorphisms ∂_m^p of the Bloch-Kato conjecture and is therefore an isomorphism. The left map is the inclusion $(\bigwedge^p F^\times \otimes \mathbb{Q}) \to \left(\bigwedge \bar{F}^\times\right)^G$. The uppermost map is induced by the projection

$$\bigwedge^p F^\times \to K_p^M(F)$$

which defines the Milnor group. To sum up: following the diagram on the right yields a surjective morphism, hence the other side has to yield a surjective morphism as well and we can conclude that $\mathrm{H}^p(X_F, \mathbb{Q}) \to \mathrm{H}^p(X_F, \mathbb{Q}/\mathbb{Z})$ has to be surjective. $\square$

This is an interesting result, as we can now say a lot more about cohomology with integral coefficients! We know now that in the long exact sequence associated to $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ all connecting homomorphisms have to be zero. Therefore we have, in every degree $p \geq 0$, an exact sequence

$$0 \to \mathrm{H}^p(X_F, \mathbb{Z}) \to \mathrm{H}^p(X_F, \mathbb{Q}) \to \mathrm{H}^p(X_F, \mathbb{Q}/\mathbb{Z}) \to 0.$$

Theorem 2.2.3. *The group* $\mathrm{H}^p(X_F, \mathbb{Z})$ *is torsion free, for every* $p \geq 0$.

2.3 On the hypothesis

It is time to face our burden: since here, we assumed that $\mathbb{Q}(\mu_\infty) \subseteq F$, which is quite an unpleasant assumption. For instance, we cannot take F to be a number field, or $\mathbb{Q}$ itself, which are obviously of primary interest. The unfortunate truth is that the construction of X_F lies on several small but crucial details, thus stretching a hypothesis usually disrupts the delicate balance on which our theorems rely. We propose a slight generalization and show why the main theorem collapses.

Let F be a characteristic 0 field (not necessarily containing all roots of unity μ_∞), let $\bar{F}/F$ be an algebraic closure and let $G = \mathrm{Gal}(\bar{F}/F)$ be the corresponding Galois group. Let's give $\bar{F}^\times$ the discrete topology and focus on the subset

$$Z_{\bar{F}} = \{\chi \in \mathrm{Hom}(\bar{F}^\times, \mathbb{S}^1) \mid \chi \text{ is injective when restricted to } \mu_\infty\} \subseteq \mathrm{Hom}(\bar{F}^\times, \mathbb{S}^1).$$

This is quite a larger subspace then $X_{\bar{F}}$. As a subspace of the Pontryagin dual $(\bar{F}^\times)^\vee = \mathrm{Hom}(\bar{F}^\times, \mathbb{S}^1)$, our set $Z_{\bar{F}}$ inherits a topology and a natural continuous left action of G given by $\sigma(\chi) = \chi \circ \sigma^{-1}$. Notice that the G-action now could modify $\chi_{|\mu_\infty}$ but our new definition allows for it. This space shares almost all nice properties of $X_{\bar{F}}$.

From the omnipresent short exact sequence $0 \to \mu_\infty \to \bar{F}^\times \to \bar{F}^\times_{\mathrm{tf}} \to 0$ we get a short exact sequence of topological groups

$$0 \to \mathrm{Hom}(\bar{F}^\times_{\mathrm{tf}}, \mathbb{S}^1) \to \mathrm{Hom}(\bar{F}^\times, \mathbb{S}^1) \to \mathrm{Hom}(\mu_\infty, \mathbb{S}^1) \to 0.$$

Notice that the first space, being isomorphic to $X_{\bar{F}}$, is connected, while the last one is totally disconnected. Hence the cosets of $X_{\bar{F}}$ in $Z_{\bar{F}}$ are precisely the connected components.

Theorem 2.3.1. *The topological space $Z_{\bar{F}}$ is non-empty, Hausdorff and compact.*

Proof. Clearly $Z_{\bar{F}}$ contains $X_{\bar{F}}$ and is therefore non-empty. Also, it is contained in the dual $(\bar{F}^\times)^\vee$, which is T_2, hence inherits this property. At last, we show that $Z_{\bar{F}}$ is a closed subspace of $\mathrm{Hom}(\bar{F}^\times, \mathbb{S}^1)$, which, being the dual of a discrete group, is compact [5, Theorem 23.17]. Take the compact set $\mu_n \subseteq \bar{F}^\times$ and the open set $U_n \subseteq \mathbb{S}^1$ obtained by carving out small closed segments around the primitive n-th roots of unity. The closed set complementary to $V(\mu_n, U_n)$ (the open set of those maps sending μ_n in U_n) is then the set of characters which sends at least an n-th root of unity to a primitive one and that are therefore injective when restricted to μ_n. We then have that

$$Z_{\bar{F}} = \bigcap_{n \in \mathbb{N}_{>1}} V(\mu_n, U_n)^\complement$$

is closed. $\square$

Alas, $Z_{\bar{F}}$ is not connected. From the aforementioned decomposition we get that $\mathrm{Hom}(\bar{F}^\times/\mu_\infty, \mathbb{S}^1)$ is the connected component of $\mathrm{Hom}(\bar{F}^\times, \mathbb{S}^1)$ containing the identity. Its cosets

$$X_{\bar{F}}(\iota) = \{\chi \in \mathrm{Hom}(\bar{F}^\times, \mathbb{S}^1) \mid \chi_{|\mu_\infty} = \iota \colon \mu_\infty \to \mathbb{S}^1\}$$

are the (homeomorphic) connected components. Hence, as $Z_{\bar{F}}$ is (as a set) the disjoint union of $X_{\bar{F}}(\iota)$ as ι varies through injective homomorphisms $\mu_\infty \to \mathbb{S}^1$, it must have infinite connected components all homeomorphic to $\mathrm{Hom}(\bar{F}^\times/\mu_\infty, \mathbb{S}^1)$. These connected components are not open, since we have an infinite number of them and $Z_{\bar{F}}$ is compact.

The G-action is still proper and free. It's free by Lemma 1.2.4, which did not require any assumption about F containing roots of unity. We need to show that the map $G \times Z_{\bar{F}} \to Z_{\bar{F}} \times Z_{\bar{F}}$ is proper but, once again, this follows trivially from compactness. We can thus define $Z_F = G \backslash Z_{\bar{F}}$ and happily conclude that being the quotient of a compact Hausdorff space by proper free action, it must be a compact Hausdorff space itself. Also, a finite extension E/F induces a covering map $Z_E \to Z_F$. This construction is far from being what we were looking for, as nor the base, nor the covering space has to be connected. Nonetheless, we can study how G acts by permutation on the connected components of $Z_{\bar{F}}$ and identify those fields for which Z_F is connected!

We saw that the connected components of $Z_{\bar{F}}$ are in one-to-one correspondence with the set of injective homomorphisms $\mu_\infty \to \mathbb{S}^1$. The image of these group homomorphisms must be composed entirely of torsion points, hence be contained in $\mu_\infty \subseteq \mathbb{S}^1$. Being injective, they are exactly the automorphisms $\mathrm{Aut}(\mu_\infty)$; as a subgroup of $(\mu_\infty)^\vee$, $\mathrm{Aut}(\mu_\infty)$ inherits the compact open topology (from a discrete copy of μ_∞ to a copy with the euclidean topology it gets as a subspace of $\mathbb{C}$). There is a (canonical) isomorphism of topological groups

$$\mathrm{Aut}(\mu_\infty) \to \mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}).$$

The group G acts on $\mathrm{Aut}(\mu_\infty)$ by composition: $\sigma(\iota) = \iota \circ \sigma^{-1}$. We are only interested on $\sigma_{|\mu_\infty}$, therefore the action factors through the quotient

$$G = \mathrm{Gal}(\bar{F}/F) \to \mathrm{Gal}(F(\mu_\infty)/F) \simeq \mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}(\mu_\infty) \cap F).$$

Thus the action on connected components is easly described as the action of the subgroup $\mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}(\mu_\infty) \cap F)$ on the group $\mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q})$ by multiplication (by the inverse, on the right).

In order to get a connect space, as the quotient by G-action, let's restrict from $Z_{\bar{F}}$ to a G-orbit of a single connected component, say $X_{\bar{F}}$. Let $Y_{\bar{F}}$ be this space. Notice that, by construction, $Y_{\bar{F}}$ is invariant for the G-action, which is proper, free and permutes transitively all connected components.

Theorem 2.3.2. *The quotient space $Y_F = G \backslash X_{\bar{F}}$ is non-empty, Hausdorff and connected. Alas, we have a non-canonical homeomorphism*

$$Y_F \to X_{F(\mu_\infty)}.$$

Proof. The first part of the statement follows, in a way that cannot be mysterious to us anymore, from the geometric properties of $Z_{\bar{F}}$ we discovered in the previous proposition and the educated behavior of the G-action. For the second part, notice that the stabilizer of each connected component is isomorphic to the open subgroup $\mathrm{Gal}(\bar{F}/F(\mu_\infty))$ of G, hence an homeomorphism

$$Y_F = G \backslash Y_{\bar{F}} \to \mathrm{Gal}(\bar{F}/F(\mu_\infty)) \backslash X_{\bar{F}} = X_{F(\mu_\infty)}.$$

□

We thus have a non-canonical homeomorphism

$$Z_F \simeq X_{F(\mu_\infty)} \times \mathrm{Gal}(\mathbb{Q}(\mu_\infty) \cap F/\mathbb{Q}).$$

This theorem tells us, sadly, that when we restrict our attention to connected spaces, we fall back in the case considered to the original article.

CHAPTER 3

Algebraic Properties

In this last chapter, we are going to discuss a construction in algebraic geometry analogous to that of X_F. The structure of this chapter is going to closely resemble that of chapter 1. The first section exploits the algebraic nature of X_F, showing this space can be retrieved as the set of rational points of a connected complex scheme $\mathcal{X}_F$. We then concentrate on the complex scheme $\mathcal{X}_{\bar{F}}$, studying its geometrical properties and showing it has no non-trivial finite étale covering space, exactly as we did for $X_{\bar{F}}$. The last section focuses on the G-action on $\mathcal{X}_{\bar{F}}$, studying its properties in order to classify the finite étale covering spaces of the quotient $\mathcal{X}_F$.

3.1 Is X_F an algebraic variety?

Our definition of $X_{\bar{F}}$ as a Pontryagin dual has an algebraic aspect we did not fully explore: it is natural to wonder if this topological space has some type of algebraic structure we were neglecting. For example, we could ask ourselves if $X_{\bar{F}}$ is a complex variety! This cannot be the case. In the first place, because this thesis would have started very differently, and, on a more serious note, because we expect $X_{\bar{F}}$ to be infinite-dimensional. Nevertheless, there is an interesting algebraic structure to discuss: that of an (infinite-dimensional) complex scheme.

Identify $\mathbb{S}^1$ with the complex unit circle and let $\iota\colon \mu_\infty \to \mathbb{S}^1 \subseteq \mathbb{C}^\times$ the embedding we fixed in the first chapter. Consider its $\mathbb{C}$-linear extension $\mathbb{C}[\iota]\colon \mathbb{C}[\mu_\infty] \to \mathbb{C}$ and the product

$$A_{\bar{F}} = \mathbb{C}[\bar{F}^\times] \otimes_{\mathbb{C}[\mu_\infty]} \mathbb{C}.$$

Call $\mathcal{X}_{\bar{F}} = \operatorname{Spec} A_{\bar{F}}$ the corresponding affine scheme. Next proposition will explain the choice of this unusual ring, clarifying its connection with X_F. Notice that the absolute Galois group G acts on $A_{\bar{F}}$, via its natural action on the first component.

Notice $\mathcal{X}_{\bar{F}}$ is a complex scheme and consider its rational points $\mathcal{X}_{\bar{F}}(\mathbb{C})$. Equip this set with the complex topology, in order to obtain a topological space $\mathcal{X}_{\bar{F}}(\mathbb{C})^{\text{top}}$.

For sake of precision, let's define the complex topology on the rational points of the spectrum $\mathcal{X} = \operatorname{Spec} A$ of a $\mathbb{C}$-algebra A (following [17]). Notice $\mathcal{X}(\mathbb{C})$ can be identified with the set of $\mathbb{C}$-algebra homomorphisms $A \to \mathbb{C}$: we can interpret an element $f \in A$ as a function $f\colon \mathcal{X}(\mathbb{C}) \to \mathbb{C}$ by sending $x\colon A \to \mathbb{C}$ to $x(f)$; we'll by writing $f(x)$ for what formally is $x(f)$. This way we can define for every $f \in A$ a subset $U_f \subseteq \mathcal{X}(\mathbb{C})$ by

$$U_f = \{x \in \mathcal{X}(\mathbb{C}) \mid |f(x)| < 1\}.$$

Definition 3.1.1. The complex topology on $\mathcal{X}(\mathbb{C})$ is the topology whose base is the family of finite intersections

$$U_{f_1,\dots,f_r} = U_{f_1} \cap \dots \cap U_{f_r}.$$

Notice that, by means of affinities on elements f_i (that is, multiplications by scalars or translations by constants), we get a lot of easily described open sets: those consisting of geometric points sending a finite number of f_i not necessarily in the open unit disc, but in any open disc of the complex plane, and thus in any open subspace $V \subseteq \mathbb{C}$. That is, all sets of the form

$$U_{V,f} = \{x \in \mathcal{X}(\mathbb{C}) \mid f(x) \subseteq V\} \subseteq \mathcal{X}(\mathbb{C}).$$

are open in complex topology.

When we put the complex topology on $\mathcal{X}_{\bar{F}}(\mathbb{C})$, we get a topological space that deformation retracts onto $X_{\bar{F}}$.

Proposition 3.1.2. *There is a canonical G-equivariant homeomorphism*

$$\mathcal{X}_{\bar{F}}(\mathbb{C})^{top} \to X_{\bar{F}} \times \operatorname{Hom}(\bar{F}^\times, \mathbb{R}).$$

Proof. This comes directly from our definitions: the space on the left hand side is

$$\mathcal{X}_{\bar{F}}(\mathbb{C}) = \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}[\bar{F}^\times] \otimes_{\mathbb{C}[\mu_\infty]} \mathbb{C}, \mathbb{C}).$$

Every map into the fiber product $\mathcal{X}_{\bar{F}}(\mathbb{C})$ can be thought as couple of maps $f\colon \mathbb{C}[\bar{F}^\times] \to \mathbb{C}$, $g\colon \mathbb{C} \to \mathbb{C}$ of $\mathbb{C}$-algebras, agreeing on roots of unity. The first map restricts to a group homomorphism $\chi_f\colon \bar{F}^\times \to \mathbb{C}^\times$, while the latter is fixed to be the identity. That is, we have a canonical bijection

$$\mathcal{X}_{\bar{F}}(\mathbb{C}) \to \{\chi \in \operatorname{Hom}(\bar{F}^\times, \mathbb{C}^\times) \mid \chi_{|\mu_\infty} = \iota\}.$$

When we consider the compact-open topology on the latter subspace of $\mathrm{Hom}(\bar{F}^\times, \mathbb{C})$ and the complex topology on $\mathcal{X}_{\bar{F}}(\mathbb{C})$, this map becomes a homeomorphism. The key observation is that in both cases open sets are exactly the set of maps that send a finite number of points to an open subspace of $\mathbb{C}$. For example, to see that the above bijection is continuous, consider an open set $V(K, U)$ of the base, where $K \subseteq \bar{F}^\times$ is compact, hence a finite set with elements $\alpha_1, \dots, \alpha_r \in F^\times$, and $U \subseteq \mathbb{C}^\times$ is an open subset. The finitely many elements of K correspond to elements $\alpha_1, \dots, \alpha_r \in A_{\bar{F}}$, hence the pre-image of $V(K, U)$ in $\mathcal{X}_{\bar{F}}(\mathbb{C})$ is an open set; having all its elements to send $\alpha_1, \dots, \alpha_r$ to the open subset $U \subseteq \mathbb{C}$. One can see that the bijection is open with an analogous argument.

Since $\mathbb{C}^\times = \mathbb{R} \times \mathbb{S}^1$ as topological groups, we get the homeomorphism

$$\{\chi \in \mathrm{Hom}(\bar{F}^\times, \mathbb{C}^\times) \mid \chi_{|\mu_\infty} = \iota\} \to \{\chi \colon \bar{F}^\times \to \mathbb{S}^1 \mid \chi_{|\mu_\infty} = \iota\} \times \mathrm{Hom}(\bar{F}^\times, \mathbb{R}).$$
$$\chi \mapsto \left(\frac{\chi}{|\chi|}, |\chi|\right).$$

Notice that the first factor is $X_{\bar{F}}$ by definition, hence the claim. Furthermore, the second factor is contractible. □

The choice of the ring $A_{\bar{F}}$ now does not look arbitrary anymore: the rational points of $\mathbb{C}[\bar{F}^\times]$ correspond to group homomorphisms $\bar{F}^\times \to \mathbb{C}^\times$ and, exactly as we did in our topological construction, we selected the component on which they agree with ι, by taking the fiber product. We can now consider the subring of G-invariants

$$(A_{\bar{F}})^G = \mathbb{C}[\bar{F}^\times]^G \otimes_{\mathbb{C}[\mu_\infty]} \mathbb{C},$$

we call this ring A_F and its spectrum $\mathcal{X}_F = \mathrm{Spec}\, A_F$. Notice that the scheme $\mathcal{X}_F$ might not be the quotient of $\mathcal{X}_{\bar{F}}$ by the G-action, because G is not necessarily finite. For the moment we content ourselves with proving that it is true for their rational points.

Proposition 3.1.3. *Let A be a $\mathbb{C}$-algebra and let G be a profinite group acting continuously on A. Let $\pi \colon \mathcal{X} = \mathrm{Spec}\, A \to \mathrm{Spec}\, A^G = \mathcal{Y}$ be the morphism associated to the inclusion $A^G \subseteq A$. Then the induced map on rational points*

$$\mathcal{X}(\mathbb{C})/G \to \mathcal{Y}(\mathbb{C})$$

is bijective.

Proof. It is well known [10, Section 2.3.4, ex 3.21] that if G is finite, then $\mathcal{X}/G \to \mathcal{Y}$ is an isomorphism and $\mathcal{X} \to \mathcal{Y}$ the topological quotient by G. Since $\mathbb{C}$ is algebraically closed, rational points are the closed points, hence the claim. We can thus assume that G is not finite.

The inclusion $A^G \subseteq A$ is integral: since the G-action is continuous, every element $x \in A$ has finite orbit $\{x_1, \dots, x_r\}$ and is solution to the monic polynomial

$$\prod_{i=1}^{r}(t - x_i) \in A^G[t].$$

Thus π is surjective and we only need to prove injectivity.

Let's write A as the filtered colimit of its subrings A^H, where $H < G$ runs through open subgroups. Let $\mathcal{Y}_H = \operatorname{Spec}(A^H)$, so that

$$\mathcal{X}(\mathbb{C}) = \varprojlim_{H<G} \mathcal{Y}_H(\mathbb{C}).$$

By the case of G finite, we know that $\mathcal{Y}_H(\mathbb{C})/(G/H) = \mathcal{Y}(\mathbb{C})$. Therefore, if $x, y \in \mathcal{X}(\mathbb{C})$ map to the same element of $\mathcal{Y}(\mathbb{C})$, then their images in $\mathcal{Y}_H(\mathbb{C})$ lie in the same G/H-orbit, in particular in the same G-orbit. For each H, we get a nonempty subset $T_H \subseteq G$ of elements which carry the image of x in $\mathcal{Y}_H(\mathbb{C})$ to the image of y in $\mathcal{Y}_H(k)$, which is open and therefore closed, because it is union of a finite number of cosets of H. The T_H form a cofiltered system of compact spaces, thus their intersection is non-empty; in this intersection lives an element of G carrying x to y. □

This allows us to carry the algebraic structure from $X_{\bar{F}}$ to X_F:

Theorem 3.1.4. *There is a deformation retraction*

$$\mathcal{X}_F(\mathbb{C})^{top} \to X_F.$$

Proof. From Theorem 3.1.2 we have a G-equivariant homeomorphism

$$\mathcal{X}_{\bar{F}}(\mathbb{C})^{\mathrm{top}} \to X_{\bar{F}} \times \operatorname{Hom}(\bar{F}^\times, \mathbb{R})$$

Consider the following deformation retract

$$\begin{aligned} H\colon X_{\bar{F}} \times \operatorname{Hom}(\bar{F}^\times, \mathbb{R}) \times [0,1] &\to X_{\bar{F}} \times \operatorname{Hom}(\bar{F}^\times, \mathbb{R}) \\ (\chi, \lambda, t) &\mapsto (\chi, t\lambda) \end{aligned}$$

The deformation retract being G-equivariant, it descends to quotients. That is, the quotient $G \backslash \mathcal{X}_{\bar{F}}(\mathbb{C})^{\mathrm{top}} = \mathcal{X}_F(\mathbb{C})^{\mathrm{top}}$ (from the just-proven Proposition 3.1.3) deformation retracts onto $G \backslash X_{\bar{F}} = X_F$. □

This exceeded our hopes, giving us an algebraic structure on all our spaces X_F and their connected coverings.

3.2 Étale covers

The previous section suggests a completely different approach to our main question, a purely algebraic one. The first step is to reformulate the question. In order to do so, we need to understand that in the realm of algebraic geometry it has been developed a Galois theory, that is not only very similar to the theories of fields extensions and covering spaces, which we recalled at the beginning of the first chapter, but encompasses both under its wide generality. We are going to refer to [19, Chapter 5] for details and present the main concepts only.

Let's start by fixing a connected scheme $\mathcal{X}$, which is going to be our base space, and define what a covering map $\mathcal{Y} \to \mathcal{X}$ is.

Definition 3.2.1. A finite étale morphism $f \colon \mathcal{Y} \to \mathcal{X}$ is a morphism such that

1. direct image sheaf $f_* \mathcal{O}_{\mathcal{Y}}$ is locally free of finite rank,
2. each fibre $\mathcal{Y}_p$ of f is the spectrum of a finite étale $\kappa(p)$-algebra.

A finite étale covering map is a surjective finite étale morphism.

As expected, a morphism of finite étale covers between $\mathcal{Y}_1, \mathcal{Y}_2 \to \mathcal{X}$ is a scheme morphism $\mathcal{Y}_1 \to \mathcal{Y}_2$ making the obvious diagram commute; if both $\mathcal{Y}_1, \mathcal{Y}_2$ are connected, any such morphism is a finite étale cover itself. In particular, we can talk about the automorphism group $\operatorname{Aut}(\mathcal{Y}|\mathcal{X})$ of a connected cover $\mathcal{Y} \to \mathcal{X}$ and consider the quotient

$$\mathcal{Y}/\operatorname{Aut}(\mathcal{Y}|\mathcal{X}) \to \mathcal{X}.$$

If this map is an isomorphism, then $\mathcal{Y}$ is said to be a Galois cover. The theory of finite étale covers develops in complete analogy with the two theories already presented.

Let $\mathcal{Y} \to \mathcal{X}$ be a Galois cover. The group $\operatorname{Aut}(\mathcal{Y}|\mathcal{X})$ determines all intermediate finite connected covers $\mathcal{Y} \to \mathcal{Z} \to \mathcal{X}$. That is, given said tower, $\mathcal{Y} \to \mathcal{Z}$ is a Galois cover whose group $\operatorname{Aut}(\mathcal{Y}|\mathcal{Z})$ is naturally identified with a subgroup $H < \operatorname{Aut}(\mathcal{Y}|\mathcal{X})$; in this case, the quotient cover $\mathcal{Y}/H \to \mathcal{X}$ is isomorphic to $\mathcal{Z} \to \mathcal{X}$. Vice versa, every subgroup $H < \operatorname{Aut}(\mathcal{Y}|\mathcal{X})$ determines an intermediate cover $\mathcal{Y} \to \mathcal{X}$, whose corresponding subgroup of $\operatorname{Aut}(\mathcal{Y}|\mathcal{X})$ is H itself.

Fix a geometric point $\bar{x}$ of $\mathcal{X}$ and consider the functor

$$Fib_{\bar{x}} \colon \mathbf{Fin\acute{E}tCov}/\mathcal{X} \to \mathbf{FinSet}$$

sending a finite étale cover $\mathcal{Y} \to \mathcal{X}$ to its geometric fiber over $\bar{x}$, which is a finite set. The group $\pi_1^{\text{ét}}(\mathcal{X}, \bar{x}) = \operatorname{Aut}(Fib_{\bar{x}})$ is called the étale fundamental group of the scheme $\mathcal{X}$ at $\bar{x}$. If we fix a Galois cover $\mathcal{Y} \to \mathcal{X}$ and restrict the fiber functor to the full subcategory of finite covers of $\mathcal{X}$ that are quotients of $\mathcal{Y}$, we obtain a finite group $\mathrm{Aut}(Fib_{\bar{x}}^{\mathcal{Y}}) = \mathrm{Aut}(\mathcal{Y}|\mathcal{X})$. The inverse system of Galois covers gives an inverse system of the corresponding automorphisms group, whose limit coincide, as an abstract group, with the étale fundamental group:

$$\pi_1^{\text{ét}}(\mathcal{X}, \bar{x}) = \mathrm{Aut}\left(Fib_{\bar{x}}\right) = \varprojlim_{\mathcal{Y}} \mathrm{Aut}(Fib_{\bar{x}}^{\mathcal{Y}}) = \varprojlim_{\mathcal{Y}} \mathrm{Aut}(\mathcal{Y}|\mathcal{X}).$$

This gives the fundamental group a profinite structure.

Theorem 3.2.2 ([19, Theorem 4.5.2])**.** *Fixed a geometric point $\bar{x} \to \mathcal{X}$, there is an equivalence*

$$Fib_{\bar{x}} \colon \textbf{\textit{FinÉtCov}}\,/\mathcal{X} \to \pi_1^{ét}(\mathcal{X}, \bar{x})\text{-}\textbf{\textit{FinSet}}$$

between the category of finite étale cover over $\mathcal{X}$ to the category of finite sets equipped with a continuous $\pi_1^{ét}(\mathcal{X}, \bar{x})$-action.

Consider the scheme $\mathcal{X} = \operatorname{Spec} F$. Its finite étale covers correspond exactly to the finite étale algebras over F; i.e. the functor Spec is an anti-equivalence between the category **FinÉtAlg** $/F$ and **FinÉtCov** $/\operatorname{Spec}(F)$. Nonetheless, it is not hard to find a natural question as interesting as the one we posed in the first chapter.

Question. Given a field F, can we find a connected scheme $\mathcal{X}_F$, defined over an algebraically closed field, whose finite covers are in correspondence with finite extensions of our base field F? That is, a space that comes equipped with an equivalence

$$\textbf{FinÉtCov}\,/\operatorname{Spec} F \to \textbf{FinÉtCov}\,/\mathcal{X}_F \quad ?$$

We already have a good candidate: the complex scheme

$$\mathcal{X}_F = \operatorname{Spec} A_F$$

we defined in the previous section. We are now going to prove this is the case, following the same steps we took in the first chapter of this thesis: the first step is to understand the geometry of $\mathcal{X}_{\bar{F}}$. In perfect analogy with the topological construction, our scheme $\mathcal{X}_{\bar{F}}$ is a connected component of the larger scheme $\operatorname{Spec} \mathbb{C}[\bar{F}^\times]$.

Proposition 3.2.3. *There is an isomorphism*

$$\mathcal{X}_{\bar{F}} \simeq \operatorname{Spec} \mathbb{C}[\bar{F}_{tf}^\times].$$

In particular, $\mathcal{X}_{\bar{F}}$ is a connected scheme.

Proof. For any abelian group M the group algebra $\mathbb{C}[M]$ is a Hopf algebra: the comultiplication $\mathbb{C}[M] \to \mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[M]$ is determined by $m \mapsto m \otimes m$ for all

$m \in M$, and the other structure maps are even more obvious. This turns $\operatorname{Spec} \mathbb{C}[M]$ into an affine commutative group scheme over $\mathbb{C}$. Consider the exact sequence

$$0 \to \mu_\infty \to \bar{F}^\times \to \bar{F}^\times_{\mathrm{tf}} \to 0, \tag{3.1}$$

and the corresponding group algebras. There is a natural way to think at $\mathbb{C}[\mu_\infty]$ as a sub-algebra of $\mathbb{C}[\bar{F}^\times]$. Since the ring homomorphism $\mathbb{C}[\mu_\infty] \hookrightarrow \mathbb{C}[\bar{F}^\times]$ is injective, the corresponding group homomorphism $\operatorname{Spec} \mathbb{C}[\bar{F}^\times] \to \operatorname{Spec} \mathbb{C}[\mu_\infty]$ is a quotient map [14, Theorem 5.43]. We can explicitly compute the kernel of this group scheme homomorphism, following [14, Chapter 1, Section e]: let $\mathbb{C}[\mu_\infty] \to \mathbb{C}$ be the map corresponding to the identity of the Hopf algebra, the one sending $m \mapsto 1$ for all $m \in \mu_\infty$. Its kernel is the augmentation ideal $I = (1 - m \mid m \in \mu_\infty) \subseteq \mathbb{C}[\mu_\infty]$, thus the group we are looking for is the spectrum of the complex algebra

$$\mathbb{C}[\bar{F}^\times]/I\mathbb{C}[\bar{F}^\times] = \mathbb{C}[\bar{F}^\times]/(1 - m \mid m \in \mu_\infty) = \mathbb{C}[\bar{F}^\times/\mu_\infty] = \mathbb{C}[\bar{F}^\times_{\mathrm{tf}}].$$

To sum up, the functor $\operatorname{Spec} \mathbb{C}[-]$ applied to (3.1) spits out the short exact sequence of affine (abelian) group schemes

$$0 \to \operatorname{Spec} \mathbb{C}[\bar{F}^\times_{\mathrm{tf}}] \to \operatorname{Spec} \mathbb{C}[\bar{F}^\times] \to \operatorname{Spec} \mathbb{C}[\mu_\infty] \to 0, \tag{3.2}$$

which we are going to prove is the short exact sequences of connected components: that is, the first scheme is the connected component of the identity, while the last is the totally disconnected quotient.

The first scheme $\operatorname{Spec} \mathbb{C}[\bar{F}^\times_{\mathrm{tf}}]$ is connected, essentially because $\bar{F}^\times_{\mathrm{tf}}$ is torsion-free: write $\bar{F}^\times_{\mathrm{tf}} = \varinjlim_n M_n$ as a direct limit over finite rank free $\mathbb{Z}$ modules $M_n \simeq \mathbb{Z}^n$, so that

$$\operatorname{Spec} \mathbb{C}[\bar{F}^\times_{\mathrm{tf}}] = \varprojlim_n \operatorname{Spec} \mathbb{C}[\mathbb{Z}^n] = \varprojlim_n (\mathbb{G}_m)^n. \tag{3.3}$$

An affine scheme is connected if and only if the corresponding ring cannot be written as a product or, equivalently, if and only if the ring has no non-trivial idempotent. That is the case: let $x \in \mathbb{C}[\bar{F}^\times_{\mathrm{tf}}]$ be an idempotent and n an integer such that $x \in \mathbb{C}[M_n]$. Since $\mathbb{C}[M_n] \simeq \mathbb{C}[\mathbb{Z}^n] \simeq \mathbb{G}^n_m$ is connected, then x must be trivial and $\operatorname{Spec} \mathbb{C}[\bar{F}^\times_{\mathrm{tf}}]$ has to be connected as well.

On the other side of the short exact sequence (3.2), the topological space underlying $\operatorname{Spec} \mathbb{C}[\mu_\infty]$ is totally disconnected: to see that, write $\mu_\infty = \varinjlim_n \mu_n$ and

$$\operatorname{Spec} \mathbb{C}[\mu_\infty] = \varprojlim_n \operatorname{Spec} \mathbb{C}[\mu_n] = \varprojlim_n \operatorname{Spec} \frac{\mathbb{C}[t]}{(t^n - 1)}.$$

The n-th object in the inverse system is the constant group scheme over $\mathbb{C}$ corresponding to $\mu_n = \mathbb{Z}/n\mathbb{Z}$, where transition maps are the obvious projections.

Getting back to the sequence (3.2), we find out that $\operatorname{Spec}\mathbb{C}[\bar{F}^\times_{\mathrm{tf}}]$ is the connected component of the identity of the group scheme $\operatorname{Spec}\mathbb{C}[\bar{F}^\times]$. Therefore, its cosets are precisely the connected components and the base change in the construction

$$\begin{array}{ccc} \mathcal{X}_{\bar{F}} & \longrightarrow & \operatorname{Spec}\mathbb{C}[\bar{F}^\times] \\ \downarrow & & \downarrow \\ \operatorname{Spec}\mathbb{C} & \longrightarrow & \operatorname{Spec}\mathbb{C}[\mu_\infty], \end{array}$$

induced by $\iota\colon \mu_\infty \to \mathbb{C}^\times$, selects the connected component we are interested in. Finally, there is a non canonical isomorphism between every two connected components, in particular the one we were seeking. □

For all properties that do not involve the G-action, we can consider the isomorphic connected component of the identity $\operatorname{Spec}\mathbb{C}[\bar{F}^\times_{\mathrm{tf}}]$. We are particularly interested in étale coverings.

Theorem 3.2.4. *$\mathcal{X}_{\bar{F}}$ has no non-trivial finite étale covering space.*

Proof. Once again, we switch to the isomorphic $\operatorname{Spec}\mathbb{C}[\bar{F}^\times_{\mathrm{tf}}]$ and consider it as the limit of complex algebraic tori in (3.3). Recall that a finite étale cover of $\operatorname{Spec}\mathbb{C}[M_n] = \operatorname{Spec}\mathbb{C}[\mathbb{Z}^n] = \mathbb{G}_m^n$ is a torus of the same dimension and the covering map is a homomorphism of tori [15, Proposition 1]: in particular, there is a free module $M \simeq \mathbb{Z}^n$ such that the finite étale cover is induced by an injective homomorphism $M_n \to M$.

Let $\mathcal{Y} \to \operatorname{Spec}\mathbb{C}[\bar{F}^\times_{\mathrm{tf}}]$ be a finite étale covering. Then $\mathcal{Y}$ must be affine, say $\mathcal{Y} = \operatorname{Spec} B$ for some finite étale ring homomorphism $\mathbb{C}[\bar{F}^\times_{\mathrm{tf}}] \to B$. By [18, Tag 00U2, item (9)], $\mathcal{Y}$ must be the base change of some finite étale cover over some space in the limit (3.3): i.e. we have a finite étale ring homomorphism $\mathbb{C}[M_n] \to B'$, for a finitely generated subgroup $\mathbb{Z}^n \simeq M_n \subseteq \bar{F}^\times_{\mathrm{tf}}$ fitting in a pull-back diagram

$$\begin{array}{ccc} \mathcal{Y} = \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} B' \\ \downarrow & & \downarrow \\ \operatorname{Spec}\mathbb{C}[\bar{F}^\times_{\mathrm{tf}}] & \longrightarrow & \operatorname{Spec}\mathbb{C}[M_n]. \end{array}$$

Notice that this very step was covered in the topological construction by the compactness argument. Since B is finite over $\mathbb{C}[\bar{F}^\times_{\mathrm{tf}}]$ so must B' be over $\mathbb{C}[M_n]$, hence $\operatorname{Spec} B' \to \operatorname{Spec}\mathbb{C}[M_n] \simeq \mathbb{G}^n$ must be a finite étale covering, therefore induced by an injective homomorphism $M_n \to M$ between free $\mathbb{Z}$-modules of the same rank n. We can pass to the dual diagram and compute

$$B = \mathbb{C}[\bar{F}^\times_{\mathrm{tf}}] \otimes_{\mathbb{C}[M_n]} \mathbb{C}[M] = \mathbb{C}\left[\bar{F}^\times_{\mathrm{tf}} \otimes_{M_n} M\right]$$

Consider the short exact sequence $0 \to M_n \to M \to Q \to 0$, taking the tensor product with the (flat) $\mathbb{Q}$-vector space $\bar{F}^\times_{\mathrm{tf}}$ kills the torsion quotient and yields an isomorphism

$$\bar{F}^\times_{\mathrm{tf}} = \bar{F}^\times_{\mathrm{tf}} \otimes_{M_n} M_n \to \bar{F}^\times_{\mathrm{tf}} \otimes_{M_n} M.$$

Therefore $B = \mathbb{C}[\bar{F}^\times_{\mathrm{tf}}]$ and $\mathcal{Y}$ is a trivial cover. □

3.3 Galois correspondence

We find ourselves in a position we are now familiar with: we have a nice space $\mathcal{X}_{\bar{F}}$ with all desirable geometric properties and no covers, on which our absolute Galois group G is acting continuously. We wish to take quotients. That is, for a finite extension E/F, we have inclusions $A^{\mathrm{Gal}(\bar{F}/F)} \subseteq A^{\mathrm{Gal}(\bar{F}/E)} \subseteq A$. Those induce a map

$$\mathcal{X}_E \to \mathcal{X}_F.$$

We wish to show this map is a finite étale covering map. In the topological analogue, this came from the G-action being free. Here, the situation is similar.

Proposition 3.3.1. *Let k be any algebraically closed field. The G-action induced on geometric points $\mathcal{X}_{\bar{F}}(k)$ is free.*

Proof. By definition

$$\mathcal{X}_{\bar{F}}(k) = \mathrm{Hom}(\mathbb{C}[\bar{F}^\times] \otimes_{\mathbb{C}[\mu_\infty]} \mathbb{C}, k),$$

the fiber product can be identified with the set of pairs (χ, g), where $\chi\colon F^\times \to k^\times$ is a group homomorphism and $g\colon \mathbb{C} \to k$ is a field embedding, such that χ and g agree on all roots of unity (notice that if k is not an extension of $\mathbb{C}$ the set $\mathcal{X}_{\bar{F}}(k)$ is empty and the statement is vacuously true). The Galois action is given by $\sigma(\chi, g) = (\sigma(\chi), g)$. The χ occurring are all injective on μ_∞ and therefore the desired result follows from the same number theory Lemma 1.2.4 we used in the topological case. □

The G action is free on geometric points. We are going to show this is exactly what we need for the quotient map to be finite étale.

Proposition 3.3.2. *Let E/F a finite extension, then $\mathcal{X}_E \to \mathcal{X}_F$ is a finite étale cover.*

The proof is quite technical. It relies on the following characterization of finite étale morphisms, stating that this property can be checked on geometric points.

Proposition 3.3.3. *Let A be a $\mathbb{C}$-algebra and G be a finite group of $\mathbb{C}$-algebra automorphisms of A. Let*

$$\pi\colon \mathcal{X} = \operatorname{Spec} A \to \mathcal{Y} = \operatorname{Spec} A^G$$

the morphism induced by the inclusion $A^G \subseteq A$.

If for every algebraically closed field k the G-action is free on geometric points $\mathcal{X}(k)$, then π is a finite étale covering.

Proof. As already mentioned, the projection π is the universal G-invariant scheme morphism from $\mathcal{X}$ and it is the topological quotient map $\mathcal{X} \to \mathcal{X}/G = \mathcal{Y}$ by the group action [10, Section 2.3.4, ex 3.21]. We are going to prove that, under our assumptions, it is finite and étale as well.

Let $G_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[G]$ be the constant group scheme associated to our group G, where $\mathbb{C}[G]$ is the commutative ring $\prod_{g\in G} \mathbb{C}$ with trivial product between different components. Consider the G-action map

$$G \times_{\mathbb{C}} \mathcal{X} \to \mathcal{X}, \quad (g, x) \mapsto gx$$

and let $\mu\colon A \to A[G]$ be the corresponding ring homomorphism. Consider also the trivial projection $G_{\mathbb{C}} \times_{\mathbb{C}} \mathcal{X} \to \mathcal{X}$ given by the inclusion

$$\delta\colon A \hookrightarrow A[G] = A \otimes_{\mathbb{C}} \mathbb{C}[G], \quad a \mapsto a \otimes 1.$$

Finally, consider the product of the just defined maps

$$\begin{aligned} R\colon G_{\mathbb{C}} \times_{\mathbb{C}} \mathcal{X} &\to \mathcal{X} \times_{\mathbb{C}} \mathcal{X} \qquad \text{corresponding to } \delta\otimes\mu\colon A \otimes_{\mathbb{C}} A \to A[G] \\ (g, x) &\mapsto (x, gx). \end{aligned} \tag{3.4}$$

The proof relies on the following two lemmas.

Lemma 3.3.4. *The product map R is a closed immersion.*

Proof. Since R is finite, because both components are, the claim follows once we prove that R is a monomorphism [18, Tag 03BB]. We check this directly: take a complex scheme $\mathcal{Z}$ and two maps

$$\mathcal{Z} \underset{(b_{\mathcal{X}}, b_G)}{\overset{(a_{\mathcal{X}}, a_G)}{\rightrightarrows}} G_{\mathbb{C}} \times_{\mathbb{C}} \mathcal{X} \xrightarrow{R} \mathcal{X} \times_{\mathbb{C}} \mathcal{X}.$$

whose compositions coincide. Taking composition with the trivial projection, we get $a_{\mathcal{X}} = b_{\mathcal{X}}$. Then we only need to check that $a_G = b_G$. Notice that both these maps have to be constant on connected components since the target scheme is totally disconnected, hence we can check if they agree on geometric points $\mathcal{Z}(k)$. The G action being free on $\mathcal{X}(k)$ means that the map $R(k) : \mathcal{X}(k) \times G(k) \to \mathcal{X}(k) \times \mathcal{X}(k)$ is injective: $(x, gx) = (y, hy)$ only if $x = y$ and $gx = hx$, which implies $g = h$ because the action is free. In particular $R(k)$ is a monomorphism of sets, hence a_G and b_G have to agree on geometric points, thus as scheme morphisms as well. □

We are interested in the coequalizer of the pair of morphisms considered so far

$$G_{\mathbb{C}} \times_{\mathbb{C}} \mathcal{X} \rightrightarrows \mathcal{X}, \qquad \text{corresponding to the equalizer of } \mu, \delta \colon A \rightrightarrows A[G].$$

This is the universal G-invariant morphism $\pi \colon \mathcal{X} \to \mathcal{Y}$ induced by $A^G \subseteq A$. Notice that, because of this universal property, relation (3.4) factors through a morphism

$$G_{\mathbb{C}} \times_{\mathbb{C}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \tag{3.5}$$

This is a closed immersion, since the composition with $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_{\mathbb{C}} \mathcal{X}$ is the closed immersion R: the dual statement on surjectivity of the corresponding ring homomorphisms is clear.

Lemma 3.3.5. *The map (3.5) is an isomorphism and A is a locally free A^G-module of constant rank n, where n is the order of G.*

Proof. Localize at a prime $p \in A^G$. Without changing notation, we assume A^G to be local. It follows that A is semi-local, i.e. it has a finite number of maximal ideals, because every point in $\mathcal{Y}$ is image through π of a single G-orbit in $\mathcal{X}$.

Since (3.5) is a closed immersion, the corresponding ring homomorphism

$$A \otimes_{A^G} A \to A[G] \tag{3.6}$$

is surjective. This means $\mu(A)$ is a generating set for $M = A[G]$ as an A-module. We claim there are $a_1, \dots, a_n \in A$ such that $\mu(a_1), \dots, \mu(a_n)$ is a basis for the free module M over A.

Let $m_1, \dots, m_r$ be the finitely many maximal ideals of A. For each maximal ideal, we can select a basis of the A/m_i-vector space M/m_iM in the image of $\mu(A)$. Via the Chinese Reminder Theorem, we can lift these basis to a generating subset $a_1, \dots, a_n \in \mu(A)$ of $M/m_1 \cap \cdots \cap m_rM$, where $n = \dim_{A/m_i} M/m_iM = \mathrm{rk}_A M$. By the Nakayama Lemma, this lifts to a set of generators of M of minimal cardinality, a basis.

The just found basis defines a map

$$\underline{a} \colon A^G \otimes_{\mathbb{C}} \mathbb{C}^n \to A, \quad 1 \otimes e_i \mapsto a_i,$$

which we claim to be an isomorphism. Consider the pair of commutative diagrams

$$\begin{tikzcd} A \otimes_{\mathbb{C}} \mathbb{C}^n \arrow[r, shift left, "\delta\otimes id"] \arrow[r, shift right, "\mu\otimes id"'] \arrow[d, "\mu(\underline{a})"] & A[G] \otimes_{\mathbb{C}} \mathbb{C}^n \arrow[d, "\mu(\underline{a})\otimes id"] \\ A[G] \arrow[r, shift left, "\delta"] \arrow[r, shift right, "m"'] & A[G \times G], \end{tikzcd}$$

where m is the map corresponding to multiplication $G \times G \to G$ and commutativity follows from the associativity property of the action. We just proved the left vertical map is an isomorphism, hence the right one

$$\mu(\underline{a}) \otimes id_{\mathbb{C}[G]} \colon \; (A \otimes_{\mathbb{C}} \mathbb{C}^n) \otimes_{\mathbb{C}} \mathbb{C}[G] \to (A \otimes_{\mathbb{C}} \mathbb{C}[G]) \otimes_{\mathbb{C}} \mathbb{C}[G]$$

is an isomorphism as well. Therefore, the induced map between their equalizers, which is $\underline{a} \colon A^G \otimes_{\mathbb{C}} \mathbb{C}^n \to A$, is an isomorphism. This proves that (3.6) is an isomorphism, because it is a map between free A-modules sending the basis $1 \otimes a_i$ to the basis $\mu(a_i)$. It follows that (3.5) is an isomorphism, since the corresponding ring homorphism, thought as a morphism of A^G-modules, is locally an isomorphism. $\square$

Since A is a locally free A^G-module, it is a flat one. In particular, $A^G \subseteq A$ is a faithfully flat ring map [18, Tag 00HQ] and the diagram

$$\begin{tikzcd} G_{\mathbb{C}} \times_{\mathbb{C}} \mathcal{X} \arrow[r, "\delta"] \arrow[d, "\mu"] & \mathcal{X} \arrow[d, "\pi"] \\ \mathcal{X} \arrow[r, "\pi"] & \mathcal{Y} \end{tikzcd}$$

is a pull-back diagram; since the left vertical map is finite, so is the right one [18, Tag 00QP]. Finally, let $p \in A^G$ a prime, consider its fiber $A \otimes_{A^G} \kappa(p)$ through π and an algebraic closure $k/\kappa(p)$. The fiber of the geometric point $\operatorname{Spec} k \to \mathcal{Y}$ has n distinct points by assumption and is the spectrum of

$$A \otimes_{A^G} \kappa(p) \otimes_{\kappa(p)} k,$$

which is an n-dimensional k-algebra by Lemma 3.3.5. It must therefore be the product k^n and $A \otimes_{A^G} \kappa(p)$ is étale, since its base change to an algebraic closure is [19, Proposition 1.5.6]. $\square$

Another step has to be taken before attacking the proof of Theorem 3.3.2. Namely, we need to prove the $\mathrm{Gal}(E/F)$-action on the geometric points of $\mathcal{X}_E$ is free.

Proposition 3.3.6. *Let k be an algebraically closed field, G a profinite group and A a $\mathbb{C}$-algebra on which G acts continuously. The projection $\pi\colon \mathcal{X} = \operatorname{Spec} A \to \operatorname{Spec} A^G = \mathcal{Y}$ induces a bijection on geometric points*

$$\mathcal{X}(k)/G \to \mathcal{Y}(k).$$

Proof. Notice that we only need to worry about fields k that are extensions of $\mathbb{C}$, otherwise $\mathcal{X}(k)$ is empty and the statement is vacuously true.

Surjectivity is easy. Since the G-action is continuous, every element $x \in A$ has finite orbit $\{x_1, \ldots, x_r\}$, hence the extension $A^G \subseteq A$ is integral: every element is solution to the monic polynomial

$$\prod_{i=1}^{r}(t - x_i) \in A^G[t].$$

Therefore π is surjective on topological points. Any geometric point $\bar{y}\colon \operatorname{Spec} k \to \mathcal{Y}$ factors through a point $y \in \mathcal{Y}$; that is

$$\operatorname{Spec} k \to \operatorname{Spec} \kappa(y) \to \mathcal{Y},$$

where the first map corresponds to the finite extension $k/\kappa(y)$. We can choose an element x in the fiber $\pi^{-1}(y)$ and consider the diagram

$$\begin{tikzcd}
& \operatorname{Spec} \kappa(x) \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\
\operatorname{Spec} k \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} \kappa(y) \arrow[r] & \mathcal{Y}.
\end{tikzcd}$$

The extension $\kappa(x)/\kappa(y)$ is finite, hence algebraic; i.e. the dotted arrow exists.

Injectivity is more involved. Namely, we need a limit argument: let's start by solving the case where G is finite.

Let $x_0, x_1 \in \mathcal{X}(k)$ be two geometric points which live in different G-orbits. This points and all their conjugates induce morphisms $g(x_i)\colon A_k = A \otimes k \to k$. Since A is an $\mathbb{C}$-algebra, this maps are all surjective, hence correspond to maximal ideals of A_k. By the Chinese Reminder Theorem, we can find an element $f \in A_k$ such that all $g(x_0)$ send $f \mapsto 0$, while all $g(x_1)$ send $f \mapsto 1$. The same holds true for the invariant element $\prod_{g \in G} g(f) \in A^G$. That is, π maps x_0 and x_1 to different geometric points of $\mathcal{Y}$.

Now, for the limit argument, let's write A as the filtered colimit of its subrings A^H, where $H < G$ runs through open subgroups. Let $\mathcal{Y}_H = \mathrm{Spec}(A^H)$, so that

$$\mathcal{X}(k) = \varprojlim_{H<G} \mathcal{Y}_H(k).$$

By the case of G finite, we know that $\mathcal{Y}_H(k)/(G/H) = \mathcal{Y}(k)$. Therefore, if $x, y \in \mathcal{X}(k)$ map to the same element of $\mathcal{Y}(k)$, then their images in $\mathcal{Y}_H(k)$ lie in the same G/H-orbit, in particular in the same G-orbit. For each H, we get a nonempty closed subset $T_H \subseteq G$ of elements which carry the image of x in $\mathcal{Y}_H(k)$ to the image of y in $\mathcal{Y}_H(k)$. The T_H form a cofiltered system of compact spaces, thus their intersection is non-empty; in this intersection lives an element of G carrying x to y. □

We can now put all the pieces together to prove the proposition.

Proof of Proposition 3.3.2. First, assume E/F is Galois. Then the map $\mathcal{X}_E \to \mathcal{X}_F$ corresponds to the inclusion

$$A_F = A_E^{\mathrm{Gal}(E/F)} \subseteq A_E.$$

Notice that for every algebraically closed field k the map $\mathcal{X}_{\bar{F}}(k) \to \mathcal{X}_E(k)$ is constant on $\mathrm{Gal}(\bar{F}/E)$-orbits, and therefore a quotient by the group action because of Proposition 3.3.6. Hence the Galois group $\mathrm{Gal}(E/F)$ acts freely on the set of geometric points $\mathcal{X}_E(k)$. By Proposition 3.3.3 this is enough to conclude.

For a general finite extension E/F, we can pass to a Galois closure E'/F. Then we have maps

$$\mathcal{X}_{E'} \to \mathcal{X}_E \to \mathcal{X}_F.$$

From what we already proved, both $\mathcal{X}_{E'} \to \mathcal{X}_F$ and $\mathcal{X}_E \to \mathcal{X}_F$ are finite étale covers, hence $\mathcal{X}_E \to \mathcal{X}_F$ has to be finite étale too. □

That assessed, we can extend our construction to finite étale algebras over F: we send a finite étale algebra $E = E_1 \times E_2 \times \cdots \times E_r$ to $\mathcal{X}_E = \mathcal{X}_{E_1} \sqcup \mathcal{X}_{E_2} \sqcup \cdots \sqcup \mathcal{X}_{E_r}$. That is, by what we just showed, a finite étale cover, although usually not connected. Morphisms of covering spaces are determined by morphisms between their connected components. The whole construction is evidently functorial.

Theorem 3.3.7. *The just defined functor*

$$\textbf{\textit{FinÉtCov}}\,/\,\mathrm{Spec}(F) \to \textbf{\textit{FinÉtCov}}\,/\mathcal{X}_F$$

is an equivalence.

Proof. The proof is substantially equivalent to the proof of the analogous Theorem 1.3.1. To prove the functor is fully faithful, it is enough to notice that Galois extension E/F are sent to Galois covers $\mathcal{X}_E \to \mathcal{X}_F$ with automorphisms group $\mathrm{Gal}(E/F)$. To prove essential surjectivity notice that

$$\varprojlim_E \mathcal{X}_E = \varprojlim_E \mathcal{X}_{\bar{F}} / \mathrm{Gal}(\bar{F}/E) = \mathcal{X}_{\bar{F}},$$

over the projective system of Galois extensions E/F. Therefore a finite étale cover $\mathcal{Y} \to \mathcal{X}_F$ pulls back through the limit to a cover $\mathcal{Y}' \to \mathcal{X}_{\bar{F}}$, which has to be trivial by Theorem 3.2.4. Trivialization must happen at a finite index E/F: that is, there is a map of étale covers $\mathcal{X}_E \to \mathcal{Y}$. This cover is Galois, hence a quotient of $\mathcal{X}_E$ by a subgroup $H < \mathrm{Aut}(\mathcal{X}_E|\mathcal{X}_F) = \mathrm{Gal}(E/F)$. This gives an isomorphism $X_{E^H} \to \mathcal{Y}$. □

Corollary 3.3.8. *Let $\bar{x} \in \mathcal{X}_F$ a character. We have an isomorphism*

$$\mathrm{Gal}(\bar{F}/F) \simeq \pi_1^{\acute{e}t}(\mathcal{X}_F, \bar{x}).$$

Proof. In the proof of the above theorem we showed a little more, namely that the equivalence

$$\mathbf{Fin\acute{E}tCov} / \mathrm{Spec}(F) \to \mathbf{Fin\acute{E}tCov} / \mathcal{X}_F$$

sends the inverse system of Galois objects over F isomorphically into the inverse system of Galois objects over $\mathcal{X}_F$. In particular

$$\begin{aligned}
\mathrm{Gal}(\bar{F}/F) &= \varprojlim_E \mathrm{Gal}(E/F) && \text{by definition}\\
&= \varprojlim_E \mathrm{Aut}(X_E|X_F) && \text{by full faithfulness}\\
&= \varprojlim_E \mathrm{Aut}(Fib_\chi^{X_E}) && \text{by Galois theory}\\
&= \mathrm{Aut}(Fib_\chi) && \text{via [19, Corollary 5.4.8]}\\
&= \pi_1^{\text{ét}}(X_F, \chi). && \text{by definition.}
\end{aligned}$$

□

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